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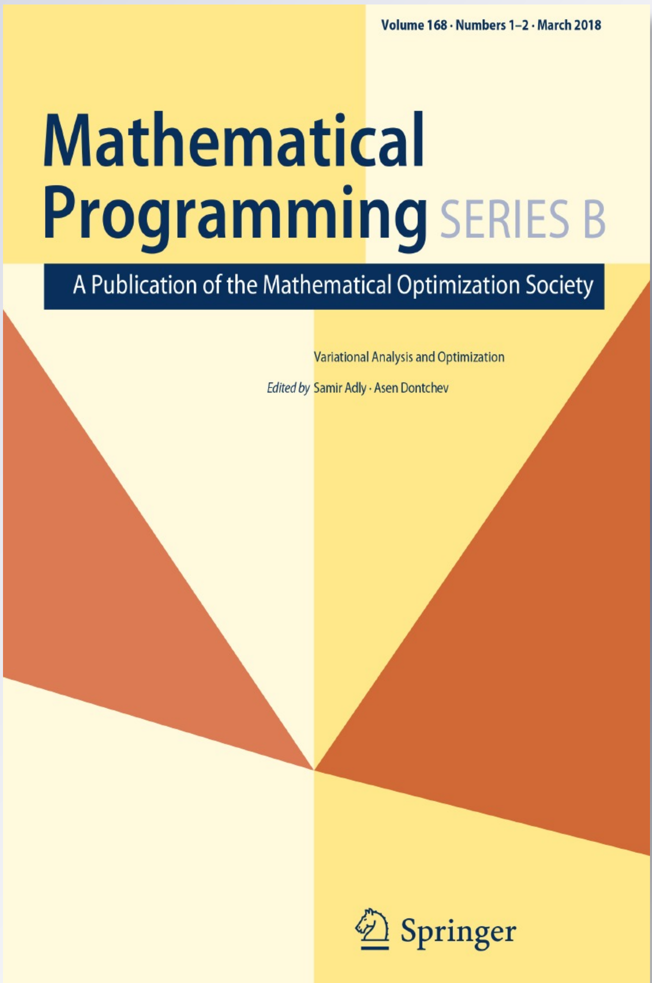

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FULL LENGTH PAPER

A structure-preserving pivotal method for affine variational inequalities

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Abstract Affine variational inequalities (AVI) are an important problem class that subsumes systems of linear equations, linear complementarity problems and optimality conditions for quadratic programs. This paper describes PATHAVI, a structure-preserving pivotal approach, that can efficiently process (solve or determine infeasible) large-scale sparse instances of the problem with theoretical guarantees and at high accuracy. PATHAVI implements a strategy known to process models with good theoretical properties without reducing the problem to specialized forms, since such reductions may destroy sparsity in the models and can lead to very long computational times. We demonstrate formally that PATHAVI implicitly follows the theoretically sound iteration paths, and can be implemented in a large scale setting using existing sparse linear algebra and linear programming techniques without employing a reduction. We also extend the class of problems that PATHAVI can process. The paper illustrates the effectiveness of our approach by comparison to the PATH solver used on a complementarity reformulation of the AVI in the context of applications in friction contact and Nash Equilibria. PATHAVI is a general purpose solver, and freely available under the same conditions as PATH.

Keywords Affine variational inequality · Normal map · Path-following algorithm

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1 Introduction

In this paper, we present PATHAVI, a structure-preserving pivotal method for affine variational inequalities (AVIs) in $\mathbb{R}^n$. An AVI(C, q, M) is defined as follows: given a polyhedral convex set C, find $z \in C$ such that

$$\langle Mz + q, y - z \rangle \geq 0, \quad \forall y \in C,$$

where $M \in \mathbb{R}^{n \times n}$, $q \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product. An AVI is a linear generalized equations [24] and we refer to [15] for results on existence, uniqueness, and stability theory for such systems.

PATHAVI tries to solve an AVI(C, q, M) by computing a zero of the normal map [25] associated with the AVI. The normal map $M_C : \mathbb{R}^n \to \mathbb{R}^n$ is defined as follows:

$$M_C(x) := M(\pi_C(x)) + q + x - \pi_C(x),$$

with $\pi_C(\cdot)$ denoting the Euclidean projector onto the set C. One can easily see that $M_C(x^*) = 0$ if and only if $z^* = \pi_C(x^*)$ where $x^* = z^* - (Mz^* + q)$ is a solution to the AVI(C, q, M). To compute a zero of $M_C(x)$, our method employs the complementary pivoting method [13,20] with a ray start: the piecewise-linear (PL) map $G_C : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is defined as

$$G_C(x, t) := M_C(x) - tr,$$

with $r \in \mathbb{R}^n$ denoting a covering vector and t an auxiliary variable. A path defined as $G_C^{-1}(0)$ is followed through complementary pivoting. The algorithm terminates when either t becomes zero (a solution to the AVI is found) or a secondary ray is generated. Under some additional assumptions this latter outcome can be interpreted in terms of the feasibility of the AVI.

The main challenge in applying the complementary pivoting method lies in the starting phase. For good theoretical properties, a ray start is required, and it is well-defined at an extreme point. However, when C contains lines there is no extreme point. To tackle this case, the previous approach [6] performs a reduction, transforming the given AVI(C, q, M) to a reduced AVI$(\tilde{C}, \tilde{q}, \tilde{M})$ to eliminate lines in C so that an extreme point is found in $\tilde{C}$, and it solves the reduced AVI. A similar approach of factoring out lines in C is used in [25, Proposition 4.1] to show a Lipschitzian homeomorphism of the normal map M_C.

A critical disadvantage of solving the reduced AVI$(\tilde{C}, \tilde{q}, \tilde{M})$ is that we may lose the original structure in C and M. The matrix $\tilde{M}$ is constructed from a Schur complement computation and the polyhedral constraints defining $\tilde{C}$ are computed by multiplying with orthonormal matrices. In particular, if the original AVI is sparse, there is no guarantee that the resulting reduced AVI would enjoy the same property. We provide an instance where this happens in Sect. 6.2. In sharp contrast, PATHAVI does not require any reduction at all. Therefore, our method is able to take advantage of a sparse structure, whereas the method in [6] often needs to perform dense linear algebra computations.

To perform a ray start in the case where there is no extreme point, PATHAVI finds an *implicit extreme point* which generalizes the notion of an extreme point when the underlying feasible region contains lines. Roughly speaking, if we project an implicit extreme point of C on the subset where all lines are removed, we obtain an extreme point. We show that there is an implicit extreme point satisfying the sufficient conditions for a ray start. We explain how phase 1 of the simplex method can be used to find such a point.

We show that PATHAVI can process an AVI(C, q, M) whenever M is an L-matrix with respect to the recession cone of C [6, Definition 4.2]. We also exhibit two new classes of AVI where PATHAVI finds a solution. The first one stems from the study of friction contact problems from an AVI perspective, and the second one can be seen as a generalization of a known existence result for LCP for copositive matrices. The conditions for the results to hold involve the whole problem data (C, q, M), whereas the previous results in [6] involve only (C, M).

A widely used method for solving an AVI is the PATH solver [9], which is considered one of the most robust and efficient solvers for mixed complementarity problems (MCPs). It is well known [11,15] that an AVI can be reformulated as a linear MCP, and PATH uses this approach when it solves an AVI. However, the MCP reformulation does not exploit the polyhedral structure of the set C, in that complementary pivoting of PATH is done over a different PL-manifold from PATHAVI's. We compare theoretical properties of the two formulations, and present computational results showing improved performance of PATHAVI.

This paper is organized as follows. In Sect. 2, we briefly describe how the complementary pivoting method on a PL-manifold computes a zero of the normal map associated with a given AVI. Section 3 presents our main theoretical results. Firstly, we discuss sufficient conditions for a ray start, define the notion of an implicit extreme point, and prove the existence of such a point satisfying the conditions for a ray start. Secondly, we show that PATHAVI can process L-matrices and introduce new types of AVIs processable by PATHAVI. In Sect. 4, we present the computational procedure to start PATHAVI. Section 5 introduces the MCP reformulation of the AVI and analyzes worst-case performance of the two formulations. We present computational results in Sect. 6 and Sect. 7 concludes this paper.

A word about our notation is in order. Let S be a convex set in $\mathbb{R}^n$. The lineality space of S is denoted by $\operatorname{lin} S$. The symbol $\operatorname{ri} S$ denotes the relative interior of S. The affine hull of S is denoted by $\operatorname{aff} S$. By $\operatorname{par} S$, we mean the subspace parallel to $\operatorname{aff} S$ such that $\operatorname{aff} S = s + \operatorname{par} S$ for each $s \in S$. The identity matrix in $\mathbb{R}^n$ is denoted by I_n and the zero vector is 0_n. When ordered index sets are used as subscripts on a matrix, they define a submatrix: for ordered index sets $\alpha \subset \{1, \ldots, m\}$ and $\beta \subset \{1, \ldots, n\}$ $M_{\alpha\beta}$ denotes a submatrix of M consisting of rows and columns of M in the order of α and β, respectively. When matrices are used as subscripts on a matrix, they define another matrix: for matrices Q and $\bar{Q}$ having appropriate dimensions $M_{Q\bar{Q}}$ denotes $Q^T M \bar{Q}$. For an AVI(C, q, M), C is assumed to be the set $\{z \in \mathbb{R}^n \mid Az - b \in K, l \leq z \leq u\}$ with $l_j, u_j \in \mathbb{R} \cup \{-\infty, \infty\}$, $b_i \in \mathbb{R}$, $A_{i\bullet} \neq 0$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, and the set K is a Cartesian product of $\mathbb{R}_+$, $\{0\}$, or $\mathbb{R}_-$ to accommodate constraints of the form $\geq$, $=$, or $\leq$, respectively. For a closed convex cone K, the dual cone of K is denoted by $K^D := \{y \mid \langle y, k\rangle \geq 0, \forall k \in K\}$. For the rest of this paper, Q and $\bar{Q}$ denote orthonormal basis matrices for the lineality space of C and its orthogonal complement, respectively.

2 Background

In this section, we briefly describe how to compute a zero of the normal map associated with a given AVI(C, q, M) using the complementary pivoting method with a ray start. We also introduce some concepts related to processability of AVIs. The reader is referred to [6,13,20,25] for more details.

The basic procedure of the complementary pivoting method to compute a zero of the normal map associated with an AVI(C, q, M) is as follows: (i) compute an initial solution (x^0, t^0) such that $G_C(x^0, t^0) = 0$, and the point (x^0, t^0) lies on a ray, called a starting ray, consisting of points $(x(t), t)$ with $G_C(x(t), t) = 0$ and $\pi_C(x(t)) = \pi_C(x^0)$ for all $t \geq t^0$; then (ii) starting from (x^0, t^0) follow a path $G_C^{-1}(0) = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}_+ \mid G_C(x, t) = 0\}$ using the complementary pivoting method until t becomes zero or a secondary ray is generated. As we will see, PATHAVI generates a starting ray at an implicit extreme point of C, i.e., $\pi_C(x^0)$ is an implicit extreme point.

Computationally, finding an initial solution (x^0, t^0) amounts to computing a complementary basic solution having $z = \pi_C(x^0)$ for the following system of equations:

$$
\begin{aligned}
Mz + q - A^T\lambda - w + v &= 0,\\
Az - b &= s,
\end{aligned} \tag{1}
$$

with complementarity between variables

$$
\begin{aligned}
K \ni s \quad &\perp \quad \lambda \in K^D,\\
0 \leq z - l \quad &\perp \quad w \geq 0,\\
0 \leq u - z \quad &\perp \quad v \geq 0.
\end{aligned} \tag{2}
$$

The complementary basic solution satisfies the sufficient conditions for a ray start as defined in Sect. 3. Then by adding $-tr$ with $r \in \operatorname{ri}(N_C(z))$ to the first equation in (1) and pivoting in the t variable, we generate an almost complementary feasible basis and start complementary pivoting.

Geometrically, the map $G_C(x, t)$ is defined over a PL$(n+1)$-manifold $\mathscr{M}_C$, where the definition of a manifold follows from [13, Section 4]. The manifold $\mathscr{M}_C$ consists of a pair $(\mathbb{R}^n \times \mathbb{R}_+, \{\sigma_i \times \mathbb{R}_+ \mid i \in \mathscr{I}\})$ such that each σ_i is a set formed by $\sigma_i = F_i + N_{F_i}$, where F_i is from a collection of the nonempty faces $\{F_i \mid i \in \mathscr{I}\}$ of C, and N_{F_i} is a normal cone having constant value on ri F_i. The manifold $\mathscr{M}_C$ is constructed from the normal manifold $\mathscr{N}_C$ consisting of a pair $(\mathbb{R}^n, \{\sigma_i \mid i \in \mathscr{I}\})$ by doing a Cartesian product each σ_i with $\mathbb{R}_+$. Note that the collection of the sets $\{\sigma_i \mid i \in \mathscr{I}\}$ is a subdivision of $\mathbb{R}^n$. Consequently, $\{\sigma_i \times \mathbb{R}_+ \mid i \in \mathscr{I}\}$ is a subdivision of $\mathbb{R}^n \times \mathbb{R}_+$. The k-dimensional faces of the $\sigma_i \times \mathbb{R}_+$ are called the k-cells of $\mathscr{M}_C$. Similarly, the

k-dimensional faces of the σ_i are called the k-cells of $\mathscr{N}_C$. The map G_C coincides with some affine transformation on each $(n+1)$-cell $\sigma_i \times \mathbb{R}_+$ as the normal map M_C does on each n-cell σ_i [25, Proposition 2.5]. Note that the starting ray $(x(t), t)$ for $t \geq t^0 > 0$ lies in the interior to some $(n+1)$-cell $\sigma_i \times \mathbb{R}_+$ of $\mathscr{M}_C$, where (x^0, t^0) is a regular point. We call a point in $\mathscr{M}_C$ a regular point if it doesn't lie in any cell $\sigma \times \tau$ of $\mathscr{M}_C$ with $\dim(G_C(\sigma \times \tau)) < n$ [13, Section 8]. Under lexicographic pivoting, each complementary pivoting generates each piece of the 1-manifold $G^{-1}(0)$ such that it starts from a boundary of a $(n+1)$-cell of $\mathscr{M}_C$ (except for the first piece containing the starting ray) and passes through the interior of that cell until it reaches a (different) boundary. If this does not occur, then we say that a secondary ray is generated. The set of $(n+1)$-cells the 1-manifold passes through never repeats [13, Lemma 15.8]. As there is a finite number of $(n+1)$-cells of $\mathscr{M}_C$, either t reaches zero (equivalently we find a solution to the AVI(C, q, M)) or a secondary ray is generated [13, Lemma 15.13].

Processability is tied to the conditions under which a secondary ray occurs. As with the LCPs, the answer to this question involves specific matrix classes that we now define.

Definition 1 (*Definition* 4.1 [6]) Let K be a closed convex cone. A matrix M is said to be *copositive* with respect to K if $\langle x, Mx\rangle \geq 0$ for all $x \in K$. If furthermore it holds that for all $x \in K$ $\langle x, Mx\rangle = 0$ implies $(M + M^T)x = 0$, then M is *copositive-plus* with respect to K.

Definition 2 Let K be a closed convex cone. A matrix M is said to be *semimonotone* with respect to K if for every $q \in \operatorname{ri}(K^D)$, the solution set of the generalized complementarity problem

$$z \in K, \qquad Mz + q \in K^D, \qquad z^T(Mz + q) = 0 \tag{3}$$

is contained in lin K.

Remark 1 This definition is consistent with the existing semimonotone property in the LCP litterature, as given in [7, Definition 3.9.1]. In this case $K = \mathbb{R}^n_+$ and lin $K = \{0\}$. Then condition (3) is equivalent to 0 being the solution set of LCP(q, M) for all $q > 0$, which by Theorem 3.9.3 in [7] is equivalent to the standard definition of M being semimonotone.

Definition 3 (*Definition* 4.2 [6]) Let K be a closed convex cone. A matrix M is said to be an *L-matrix* with respect to K if both

(a) M is semimonotone with respect to K
(b) For any $z \neq 0$ satisfying

$$z \in K, \qquad Mz \in K^D, \qquad z^T Mz = 0,$$

there exists $z' \neq 0$ such that z' is contained in every face of K containing z and $-M^T z'$ is contained in every face of K^D containing Mz.

Lemma 1 (Lemma 4.5 [6]) *If a matrix M is copositive-plus with respect to a closed convex cone K, then it is an L-matrix with respect to K.*

The main existing result on the processability using a path following method is the following.

Theorem 1 (Theorem 4.4 [6]) *Suppose that C is a polyhedral convex set, and M is an L-matrix with respect to* rec C *which is invertible on the lineality space of C. Then exactly one of the following occurs:*

- *The method of* [6] *solves the AVI(C, q, M).*
- *The following system has no solution*

$$Mz + q \in (\text{rec } C)^D.$$

3 Theoretical results

In this section, we show that an implementation of PATHAVI in the original space enjoys the same properties as Theorem 1. We first identify sufficient conditions to allow a ray start. We define an *implicit extreme point*, which is a generalization of an extreme point when the lineality space is nontrivial, and show that there exists an implicit extreme point satisfying these sufficient conditions. A computational method for finding such an implicit extreme point is described in Sect. 4. Our conditions generalize those required for existing pivotal methods [6,9,20] for LCP, MCP, and AVI.

PATHAVI can process L-matrices with respect to the recession cone of the feasible set of the AVI. To this end, we show that a 1-manifold [the path $G_C^{-1}(0)$] generated by PATHAVI with a ray start at an implicit extreme point corresponds to a 1-manifold generated by the same pivotal method with a ray start at an extreme point in the reduced space. The reduced space is formed by projecting out the lineality space. This one-to-one correspondence is derived from the structural correspondence of the faces and the normal cones between the original space and the reduced one. Then by applying the existing processability result to the 1-manifold in the reduced space, we obtain the desired result.

3.1 Sufficient conditions for a ray start and processability of PATHAVI

We first identify sufficient conditions to perform a ray start at a point.

Proposition 1 *Let an AVI(C, q, M) be given. If the following conditions are satisfied at a point $\bar{z} \in C$ with $\bar{z} + \text{lin } C$ being a face of C, then we can perform a ray start at $\bar{z}$.*

- $M\bar{z} + q \in \text{aff } (N_C(\bar{z}))$.
- *Every point in the interior of the $(n+1)$-cell $((\bar{z} + \text{lin } C) + N_C(\bar{z})) \times \mathbb{R}_+$ is regular. (See Sect. 2 for the definition of a regular point.)*

- *There exists a complementary basis at* $\bar{z}$ *such that* aff $(N_C(\bar{z}))$ *is spanned by columns of the basic variables in* (λ, w, v).

Proof Pick a vector $r \in \text{ri}\,(N_C(\bar{z}))$. Let (z, λ, w, v, s) be the complementary basic solution to (1) and (2) corresponding to the given complementary basis. Note that $z = \bar{z}$, thus s is feasible. Therefore, only basic variables in (λ, w, v) might be infeasible. The first and third conditions say that we have $Mz + q - A^T\lambda - w + v = 0$. By the third condition, for each $t \geq 0$ we have a unique $(\lambda(t), w(t), v(t))$ satisfying $Mz + q - A^T\lambda(t) - w(t) + v(t) - tr = 0$. As $r \in \text{ri}\,(N_C(\bar{z}))$, there exists $t^0 \geq 0$ such that for all $t \geq t^0$ we have $Mz+q-A^T\lambda(t)-w(t)+v(t)-tr = 0$ and $(\lambda(t), w(t), v(t))$ are feasible variables. Then for all $t \geq t^0$ $(x(t), t)$ with $x(t) := \bar{z}-A^T\lambda(t)-w(t)+v(t)$ lies in the cell $((\bar{z} + \text{lin}\ C) + N_C(\bar{z})) \times \mathbb{R}_+$ with $\pi_C(x(t)) = \bar{z}$ and $G_C(x(t), t) = 0$. By the second condition, the ray $(x(t), t)$ is generated at a regular point. By pivoting the t variable into the complementary basis, we see that we can perform a ray start at $\bar{z}$. □

Note that the sufficient conditions are satisfied at an extreme point. If z is an extreme point, then aff $(N_C(z)) \equiv \mathbb{R}^n$ thus the first condition is trivially satisfied. Each extreme point has a corresponding basic feasible solution (BFS) to $Ax - b = s$ [23, Section 3.4], and with that BFS we can construct a complementary basis satisfying the third condition as shown in Proposition 10 later in this paper. The second condition is also satisfied as proved in Proposition 4. As the existing pivotal methods [6,9,20] for LCP, MCP, and AVI perform a ray start at an extreme point, we see that the sufficient conditions generalize the existing result.

We now define an implicit extreme point, which is a generalization of an extreme point when the lineality space is nontrivial.

Definition 4 Let C be a convex set in $\mathbb{R}^n$. A point $z \in C$ is called *an implicit extreme point* of C if $z = \lambda z^1 + (1 - \lambda)z^2$ for any $z^1, z^2 \in C$ and $\lambda \in (0, 1)$ implies that $z - z^1 \in \text{lin}\ C$ and $z - z^2 \in \text{lin}\ C$.

Note that if the lineality space of C is trivial, that is, lin $C = \{0\}$, then the definition of an implicit extreme point coincides with definition of an extreme point.

In the following four propositions, we provide some properties of implicit extreme points, which are generalization of the ones enjoyed by extreme points. They are used as a tool for showing the existence of an implicit extreme point satisfying the sufficient conditions and for structural analysis later in this section. We start with faces consisting of only implicit extreme points. This generalizes 0-dimensional faces that are equivalent to extreme points. As the proof is elementary, we omit it.

Proposition 2 *Let C be a nonempty convex set in $\mathbb{R}^n$ and $\ell = \dim(\text{lin}\ C)$. Then every point in an ℓ-dimensional face of C is an implicit extreme point of C. Also, for each implicit extreme point z of C we have $F = z + \text{lin}\ C$ is an ℓ-dimensional face of C.*

We next prove that the affine hull of the normal cone to C at an implicit extreme point is the orthogonal complement of the lineality space of C. This generalizes the fact that the normal cone to C at an extreme point is full-dimensional.

Proposition 3 *A point $z \in C$ is an implicit extreme point of a nonempty polyhedral convex set C in $\mathbb{R}^n$ if and only if* $\text{aff}\,(N_C(z)) = (\text{lin}\ C)^{\perp}$.

Proof (only-if) Suppose that z is an implicit extreme point of C. Using Proposition 2, $F = z + \text{lin}\ C$ is a face of C. We then have $\text{par}\ F = \text{lin}\ C$. By [25, Proposition 2.1], $\text{par}\ F = (\text{aff}\ N_F)^{\perp}$, where N_F represents the normal cone having the same value for all $\hat{z} \in \text{ri}\ F$, i.e., $N_C(\hat{z}^1) = N_F = N_C(\hat{z}^2)$ for all $\hat{z}^1, \hat{z}^2 \in \text{ri}\ F$. As $z \in \text{ri}\ F$, it follows that $\text{aff}\,(N_C(z)) = (\text{lin}\ C)^{\perp}$.

(if) Suppose that $z \in C$ and $\text{aff}\,(N_C(z)) = (\text{lin}\ C)^{\perp}$. Pick a face F of C such that $z \in \text{ri}\ F$. Such a face exists by [27, Theorem 18.2]. Then $N_C(z) = N_F$, where N_F is the normal cone having constant value on $\text{ri}\ F$. As $\text{par}\ F = (\text{aff}\ N_F)^{\perp}$, we then have $\text{par}\ F = \text{lin}\ C$ and $F = z + \text{lin}\ C$. By Proposition 2, z is an implicit extreme point of C. □

Next we show that the second condition in Proposition 1 is satisfied at an implicit extreme point. Note that in the proposition below we show $\dim(M_C(\sigma)) = n$, which implies that $\dim(G_C(\sigma \times \mathbb{R}_+)) = n$.

Proposition 4 *Let z be an implicit extreme point of a nonempty polyhedral convex set C in $\mathbb{R}^n$ and σ be the cell $((z + \text{lin}\ C) + N_C(z))$ in the normal manifold of C. Then for an AVI(C, q, M) with M invertible on the lineality space of C, we have* $\dim(M_C(\sigma)) = n$.

Proof By [25, Proposition 2.5], M_C coincides with some affine transformation A_σ on σ. In the basis $Z = (Q\ \ \bar{Q})$, we can represent the matrix $A_\sigma(\cdot) - A_\sigma(z)$ as follows:

$$\begin{bmatrix} Q^T M Q & 0 \\ \bar{Q}^T M Q & I \end{bmatrix}.$$

As $Q^T M Q$ is invertible, the matrix $A_\sigma(\cdot) - A_\sigma(z)$ is invertible. As σ is n-dimensional, the result follows. □

Finally, let us consider a ℓ-dimensional face F with $\ell = \dim(\text{lin}\ C)$ (hence consisting of only implicit extreme points by Proposition 2). Then there exists an implicit extreme point $z \in F$ such that $Mz + q \in \text{aff}\,(N_C(z))$. This generalizes the fact that at each extreme point $\bar{z}$ we have $M\bar{z} + q \in \text{aff}\,(N_C(\bar{z})) = \mathbb{R}^n$.

Proposition 5 *Let an AVI(C, q, M) problem be given and $z \in C$ be an implicit extreme point of C. Assume that M is invertible on the lineality space of C. Then there exists $\hat{z} \in z + \text{lin}\ C$ such that* $M\hat{z} + q \in \text{aff}\,(N_C(\hat{z}))$.

Proof For any implicit extreme point $\hat{z}$ of C, $M\hat{z} + q \in \text{aff}\,(N_C(\hat{z}))$ if and only if $\pi_{\text{lin}\ C}(M\hat{z} + q) = 0$ by Proposition 3. By the assumption, M_{QQ} is invertible. Set

$$\hat{z} = z + Qy \quad \text{where} \quad y = -M_{QQ}^{-1}(Q^T q + M_{Q\bar{Q}}\bar{Q}^T z) - Q^T z.$$

Then $\hat{z} \in z + \text{lin}\ C$ thus $\hat{z}$ is an implicit extreme point of C by Proposition 2, and

$$
\begin{aligned}
Q^T(M\hat{z}+q) &= Q^T\left(M\left[Q\ \bar{Q}\right]\begin{bmatrix}Q^T\\ \bar{Q}^T\end{bmatrix}\hat{z}+q\right),\\
&= M_{QQ}\left(Q^T\hat{z}\right)+M_{Q\bar{Q}}\left(\bar{Q}^T\hat{z}\right)+Q^Tq,\\
&= M_{QQ}\left(Q^Tz+y\right)+M_{Q\bar{Q}}\left(\bar{Q}^Tz\right)+Q^Tq,\\
&= 0.
\end{aligned}
$$

It follows that $\pi_{\text{lin}\ C}(M\hat{z}+q)=0$. □

By Propositions 4 and 5, there exists an implicit extreme point satisfying the first two sufficient conditions for a ray start. We postpone checking the third condition to Sect. 4 as it requires a constructive proof. For the rest of this section, we assume that we have an implicit extreme point satisfying the sufficient conditions.

We now turn our attention to the processability of PATHAVI. Assume that we perform a ray start at an implicit extreme point and generate a 1-manifold in the original space $\mathbb{R}^n$. Our basic idea of deriving processability is that this 1-manifold corresponds to a 1-manifold generated by the same pivotal method with a ray start at an extreme point defined in the reduced space having possibly smaller dimension. We can then apply the existing processability result [6, Theorem 4.4]. To establish the correspondence, we prove that there is a one-to-one correspondence between the faces, the normal cones, and the full-dimensional cells of the original space and reduced space.

Proposition 6 *Let C be a nonempty polyhedral convex set in $\mathbb{R}^n$ and $\tilde{C}$ be the set $\tilde{C}=\bar{Q}^TC=\{\tilde{z}\,|\,\tilde{z}=\bar{Q}^Tz$ for some $z\in C\}$ defined in $\mathbb{R}^{n-\ell}$ where $\ell=\dim(\text{lin}\ C)$. Then the followings hold.*

- (a) *z is an implicit extreme point of C if and only if $\tilde{z}=\bar{Q}^Tz$ is an extreme point of $\tilde{C}$.*
- (b) *F is a face of C if and only if $\tilde{F}=\bar{Q}^TF$ is a face of $\tilde{C}$.*
- (c) *$v\in N_C(z)$ if and only if $v=\bar{Q}\tilde{v}$ for some $\tilde{v}\in N_{\tilde{C}}(\tilde{z})$ where $\tilde{z}=\bar{Q}^Tz$.*
- (d) *σ is an n-cell of the normal manifold $\mathcal{N}_C$ of C if and only if $\tilde{\sigma}=\bar{Q}^T\sigma$ is an $(n-\ell)$-cell of the normal manifold $\mathcal{N}_{\tilde{C}}$.*

Proof We prove in sequence. (a) (only-if) Let z be an implicit extreme point of C. Set $\tilde{z}=\bar{Q}^Tz$. We prove by contradiction. Suppose that $\exists\tilde{z}^1,\tilde{z}^2\in\tilde{C}$ and $\lambda\in(0,1)$ such that $\tilde{z}=\lambda\tilde{z}^1+(1-\lambda)\tilde{z}^2$ with $\tilde{z}\neq\tilde{z}^i$ for $i=1,2$. By definition of $\tilde{C}$, we have $z^1,z^2\in C$ such that $\tilde{z}^i=\bar{Q}^Tz^i$ for $i=1,2$. As $C=\text{lin}\ C\oplus((\text{lin}\ C)^{\perp}\cap C)$ [27, 65] and $\bar{Q}^Tz=\bar{Q}^T(\lambda z^1+(1-\lambda)z^2)$, there exists $a\in\text{lin}\ C$ such that $z=\lambda(a+z^1)+(1-\lambda)(a+z^2)$. As $\bar{Q}^T(z-(a+z^i))=\tilde{z}-\tilde{z}^i\neq 0$, we have $z-(a+z^i)\notin\text{lin}\ C$ for $i=1,2$, which contradicts our assumption that z is an implicit extreme point of C.

(if) Using similar proof technique, we can show that for an extreme point $\tilde{z}\in\tilde{C}$, z is an implicit extreme point of C when $\tilde{z}=\bar{Q}^Tz$.

(b) (only-if) Let F be a face of C. Set $\tilde{F} = \bar{Q}^T F$. Clearly, $\tilde{F}$ is a convex subset of $\tilde{C}$. Let $\tilde{z}^1, \tilde{z}^2 \in \tilde{C}$ and $\lambda \in (0, 1)$ satisfying $\lambda\tilde{z}^1 + (1-\lambda)\tilde{z}^2 \in \tilde{F}$. From $C = \text{lin } C \oplus ((\text{lin } C)^\perp \cap C)$, we have $\bar{Q}\tilde{z}^i \in C$ for $i = 1, 2$. Then $\bar{Q}(\lambda\tilde{z}^1 + (1-\lambda)\tilde{z}^2) \in F$ so that $\bar{Q}\tilde{z}^1 \in F$ and $\bar{Q}\tilde{z}^2 \in F$. This shows that $\tilde{z}^i \in \tilde{F}$ for $i = 1, 2$.

(if) Let $\tilde{F} = \bar{Q}^T F$ be a face of $\tilde{C}$. By the definition of $\tilde{F}$, F is a convex subset of C. Let $z^1, z^2 \in C$ and $\lambda \in (0, 1)$ such that $\lambda z^1 + (1-\lambda)z^2 \in F$. We have $\bar{Q}^T z^i \in \tilde{C}$ for $i = 1, 2$ and $\bar{Q}^T(\lambda z^1 + (1-\lambda)z^2) \in \tilde{F}$. Thus, $\bar{Q}^T z^i \in \tilde{F}$, hence $z^i \in F + \text{lin } C$ for $i = 1, 2$. Therefore, $z^i \in F$ for $i = 1, 2$.

(c) For a vector $v \in \mathbb{R}^n$, we represent components of v in lin C and $(\text{lin } C)^\perp$ in the basis $\begin{bmatrix} Q & \bar{Q} \end{bmatrix}$ by v_Q and $v_{\bar{Q}}$, respectively, so that $v = Qv_Q + \bar{Q}v_{\bar{Q}}$. If either $z \notin C$ or $\tilde{z} \notin \tilde{C}$, then we have nothing to prove. Therefore, we assume that $z \in C$ and $\tilde{z} \in \tilde{C}$ in the proof. (only-if) Let $v \in N_C(z)$. By the definition of the normal cone, for each $a \in \text{lin } C$ we have $\langle v, (z+a) - z\rangle \le 0$ and $\langle v, (z-a) - z\rangle \le 0$. Thus, $\langle v, a\rangle = 0$ for all $a \in \text{lin } C$. Whence $N_C(z) \subset (\text{lin } C)^\perp$ so that $v_Q = 0$ and $v = \bar{Q}v_{\bar{Q}}$. We then have

$$\begin{aligned} 0 &\ge \langle v, y - z\rangle, \quad \forall y \in C \\ &= \left\langle \bar{Q}v_{\bar{Q}}, Qy_Q + \bar{Q}y_{\bar{Q}} - (Qz_Q + \bar{Q}z_{\bar{Q}})\right\rangle \\ &= \left\langle \bar{Q}v_{\bar{Q}}, \bar{Q}(y_{\bar{Q}} - z_{\bar{Q}})\right\rangle \\ &= \langle v_{\bar{Q}}, y_{\bar{Q}} - z_{\bar{Q}}\rangle \end{aligned}$$

By setting $\tilde{v} = v_{\bar{Q}}$, $v = \bar{Q}\tilde{v}$ and $\tilde{v} \in N_{\tilde{C}}(\tilde{z})$.

(if) Let $\tilde{v} \in N_{\tilde{C}}(\tilde{z})$ and set $v = \bar{Q}\tilde{v}$. We have $\tilde{z} = \bar{Q}^T z$ if and only if $z \in \text{lin } C + \bar{Q}\tilde{z}$. Let $z \in \text{lin } C + \bar{Q}\tilde{z}$. Then

$$\langle v, y - z\rangle = \left\langle \tilde{v}, y_{\bar{Q}} - z_{\bar{Q}}\right\rangle \le 0, \quad y \in C$$

and the result follows.

(d) The conclusion follows from (b), (c), and the definition of the full-dimensional cells of the normal manifold. □

A similar result holds for the 1-manifold $G_C^{-1}(0)$.

Proposition 7 *Let an AVI(C, q, M) problem be given. Suppose that the matrix M is invertible on the lineality space of C, and $G_C(x^*, t^*) = 0$ with $r \in N_C(\pi_C(x^0))$ for some $x^0 \in \mathbb{R}^n$. Then the PL function $\tilde{G}_{\tilde{C}}(\tilde{x}, t) := \tilde{M}\pi_{\tilde{C}}(\tilde{x}) + \tilde{q} + \tilde{x} - \pi_{\tilde{C}}(\tilde{x}) - t\tilde{r}$ has value zero at $(\tilde{x}^*, t^*)$, where*

$$\begin{aligned} \tilde{x}^* &= \bar{Q}^T x^* \\ Z &= \begin{bmatrix} Q & \bar{Q} \end{bmatrix}, \\ \tilde{M} &= \left(Z^T M Z / M_{QQ}\right) = M_{\bar{Q}\bar{Q}} - M_{\bar{Q}Q}M_{QQ}^{-1}M_{Q\bar{Q}}, \\ \tilde{C} &= \bar{Q}^T C,\ \tilde{x}^0 = \bar{Q}^T x^0,\ \tilde{q} = (\bar{Q}^T - M_{\bar{Q}Q}M_{QQ}^{-1}Q^T)q, \\ \tilde{r} &= \bar{Q}^T r \in N_{\tilde{C}}\left(\pi_{\tilde{C}}(\tilde{x}^0)\right). \end{aligned}$$

Conversely, if $\tilde{G}_{\tilde{C}}(\tilde{x}^*, t^*) = 0$ *then* $G_C(x^*, t^*) = 0$ *with* $x^* = \bar{Q}\tilde{x}^* + Qy^*$ *and*

$$y^* = -M_{QQ}^{-1}\left(M_{Q\bar{Q}}\pi_{\tilde{C}}(\tilde{x}^*) + Q^T q\right).$$

Proof Let (x^*, t^*) satisfying $G_C(x^*, t^*) = 0$ with $r \in N_C(\pi_C(x^0))$ for some x^0 be given. Then

$$\begin{aligned}
&M\pi_C(x^*) + q + x^* - \pi_C(x^*) - t^* r = 0,\\
(\Rightarrow)\ &\begin{bmatrix} Q^T \\ \bar{Q}^T \end{bmatrix} M \begin{bmatrix} Q & \bar{Q} \end{bmatrix} \begin{bmatrix} Q^T \\ \bar{Q}^T \end{bmatrix} \pi_C(x^*) + \begin{bmatrix} Q^T \\ \bar{Q}^T \end{bmatrix} (q + x^* - \pi_C(x^*) - t^* r) = 0,\\
(\Rightarrow)\ &\begin{bmatrix} M_{QQ} & M_{Q\bar{Q}} \\ M_{\bar{Q}Q} & M_{\bar{Q}\bar{Q}} \end{bmatrix} \begin{bmatrix} Q^T \pi_C(x^*) \\ \bar{Q}^T \pi_C(x^*) \end{bmatrix} + \begin{bmatrix} Q^T q \\ \bar{Q}^T (q + x^* - \pi_C(x^*) - t^* r) \end{bmatrix} = 0,\\
(\Rightarrow)\ &\tilde{M}\bar{Q}^T \pi_C(x^*) + \tilde{q} + \bar{Q}^T(x^* - \pi_C(x^*)) - t^*\tilde{r} = 0,\\
&\quad \text{using } Q^T\pi_C(x^*) = -M_{QQ}^{-1}(M_{Q\bar{Q}}\bar{Q}^T\pi_C(x^*) + Q^T q),\\
(\Rightarrow)\ &\tilde{M}\pi_{\tilde{C}}(\tilde{x}^*) + \tilde{q} + \tilde{x}^* - \pi_{\tilde{C}}(\tilde{x}^*) - t^*\tilde{r} = 0.
\end{aligned}$$

The second $(\Rightarrow)$ holds because $N_C(z) \subset (\text{lin } C)^\perp, \forall z \in C$ as shown in the proof of Proposition 6(c). The last $(\Rightarrow)$ holds because $\bar{Q}^T\pi_C(x) = \pi_{\bar{Q}^T C}(\bar{Q}^T x)$ by [5, Lemma 2.1]. Also, $\tilde{r} \in N_{\tilde{C}}(\pi_{\tilde{C}}(\tilde{x}^0))$ by Proposition 6(d).

Conversely, let $\tilde{G}_{\tilde{C}}(\tilde{x}^*, t^*) = 0$. Set $x^* = \bar{Q}\tilde{x}^* + Qy^*$ with y^* as specified in the proposition. Then

$$\begin{aligned}
\pi_C(x^*) &= \pi_{C\cap(\text{lin } C)^\perp}(x^*) + \pi_{\text{lin } C}(x^*)\\
&= \bar{Q}\pi_{\tilde{C}}(\tilde{x}^*) + Qy^*.
\end{aligned}$$

Therefore, $Q^T\pi_C(x^*) = y^*$. By the definition of y^*, all the converse directions also hold. □

Note that the AVI$(\tilde{C}, \tilde{q}, \tilde{M})$ with $\tilde{C}, \tilde{q}$, and $\tilde{M}$ as in Proposition 7 is the same problem obtained by applying the stage 1 reduction [6, 49] to the AVI(C, q, M). Also, $\tilde{G}_{\tilde{C}}$ is the PL function defined on the $(n - \dim(\text{lin } C) + 1)$-manifold $\mathscr{M}_{\tilde{C}}$ of $\tilde{C}$ to find a zero of the normal map associated with the AVI$(\tilde{C}, \tilde{q}, \tilde{M})$.

An implication of Proposition 7 is that if $G_C(x + \theta\Delta x, t + \theta\Delta t) = 0$ with $(x + \theta\Delta x, t + \theta\Delta t) \in \sigma \times \mathbb{R}_+$ for all $\theta \in [0, \nu]$ with $\nu > 0$ (possibly $\nu = \infty$) and $\sigma \times \mathbb{R}_+$ is an $(n+1)$-cell of $\mathscr{M}_C$, then we have $\tilde{G}_{\tilde{C}}(\tilde{x} + \theta\Delta\tilde{x}, t + \theta\Delta t) = 0$ with $(\tilde{x} + \theta\Delta\tilde{x}, t + \theta\Delta t) \in \tilde{\sigma} \times \mathbb{R}_+$ for all $\theta \in [0, \nu]$, where $\tilde{\sigma} = \bar{Q}^T\sigma$ and $\Delta\tilde{x} = \bar{Q}^T\Delta x$. The converse also holds by setting $\Delta x = \bar{Q}\Delta\tilde{x} + Q\Delta y$ with $\Delta y = -M_{QQ}^{-1}M_{Q\bar{Q}}H_{\tilde{\sigma}}\Delta\tilde{x}$, where $H_{\tilde{\sigma}}$ is an orthogonal projector onto par $\tilde{F}$ with $\tilde{\sigma} = \tilde{F} + N_{\tilde{F}}$ [25, see the proof of Proposition 2.5]. Therefore, the projection of each piece of $G_C^{-1}(0)$ onto $\mathscr{M}_{\tilde{C}}$ corresponds to each piece of $\tilde{G}_{\tilde{C}}^{-1}(0)$ and vice versa. As a consequence, if $G_C^{-1}(0)$ contains a ray, i.e., there exists $(\Delta x, \Delta t) \neq 0$ with $\nu = \infty$ on some $(n + 1)$-cell $\sigma \times \mathbb{R}_+$ of $\mathscr{M}_C$, and the corresponding value $(\Delta\tilde{x}, \Delta t)$ is not zero, then the corresponding piece of $\tilde{G}_{\tilde{C}}^{-1}(0)$ is also a ray. The following proposition shows that whenever there is a ray in $G_C^{-1}(0)$ with $\Delta x \neq 0$, then we have $\Delta\tilde{x} \neq 0$ so that the corresponding piece of $\tilde{G}_{\tilde{C}}^{-1}(0)$ is also a ray. Note that the converse automatically holds as $\Delta x \neq 0$ for each $\Delta\tilde{x} \neq 0$.

Proposition 8 *For an AVI*(C, q, M)*, suppose that* PATHAVI *generates* $G_C^{-1}(0)$ *with a ray start at an implicit extreme point. For each ray in* $G_C^{-1}(0)$ *in the direction of* $(\Delta x, \Delta t) \neq 0$*, if* $\Delta x \neq 0$ *then* $\Delta\tilde{x} := \bar{Q}^T \Delta x$ *is a ray in* $\tilde{G}_{\tilde{C}}^{-1}(0)$ *that is nonzero under the assumption that either* 0 *is a regular value or we perform lexicographic pivoting.*

Proof Let z be an implicit extreme point at which PATHAVI performs a ray start. By construction, $\Delta\tilde{x} = 0$ if and only if $\Delta x \in \operatorname{lin} C$. For the starting ray we have $\Delta x \in N_C(z)$ so that $\Delta x \notin \operatorname{lin} C$ by Proposition 3. Thus, $\Delta\tilde{x} \neq 0$.

We now assume that there is a ray in $G_C^{-1}(0)$ other than the starting ray. Suppose that $\Delta x \in \operatorname{lin} C$. We proceed by contradiction. Let us assume that the ray is generated at the $(k+1)$th iteration of complementary pivoting, and that it starts from (x^{k+1}, t^{k+1}). We know that $(x^{k+1}, t^{k+1}) \in (\sigma^{k+1} \times \mathbb{R}_+) \cap (\sigma^k \times \mathbb{R}_+)$, where $\sigma^k \times \mathbb{R}_+$ is the $(n+1)$-cell of $\mathscr{M}_C$ PATHAVI passes through at the kth complementary pivoting iteration. As $\operatorname{lin} C \subset \operatorname{lin} \sigma$ for each $(n+1)$-cell $\sigma \times \mathbb{R}_+$ of $\mathscr{M}_C$, $x^{k+1} + \theta\Delta x \in \sigma^k$ for all $\theta \geq 0$. This contradicts the fact that $G_C^{-1}(0)$ is a 1-manifold neat in $\mathscr{M}_C$ [13, Theorem 9.1 or Lemma 15.5], that is, $G^{-1}(0) \cap (\sigma^k \times \mathbb{R}_+)$ must be expressed as an intersection of $\sigma^k \times \mathbb{R}_+$ with a line. Therefore, $\Delta x \notin \operatorname{lin} C$ and the result follows. □

From Lemma 5 (in the Appendix), if M is semimonotone with respect to rec C and invertible on lin C, we have $\Delta t = 0$ whenever PATHAVI generates a ray in the direction of $(\Delta x, \Delta t)$. Matrix classes having the property $\Delta t = 0$ include the L-matrix class and the new matrix classes defined in Sect. 3.2. Therefore, whenever PATHAVI generates a ray in $G_C^{-1}(0)$ for those classes of matrices the corresponding piece in $\tilde{G}_{\tilde{C}}^{-1}(0)$ is also a ray by Proposition 8.

Equipped with Propositions 6–8, we now show that PATHAVI can process L-matrices. In contrast to [6], we do not resort to a reduction to show the result.

Theorem 2 *Suppose that C is a polyhedral convex set, and M is an L-matrix with respect to* rec C *which is invertible on the lineality space of C. Then exactly one of the following occurs:*

- PATHAVI *solves the AVI*(C, q, M).
- *The following system has no solution*

$$Mz + q \in (\operatorname{rec} C)^D.$$

Proof By Propositions 6–7, for a 1-manifold $G_C^{-1}(0)$ generated by PATHAVI there corresponds to a 1-manifold $\tilde{G}_{\tilde{C}}^{-1}(0)$ in the reduced space generated by the same pivotal method with a ray start at an extreme point of $\tilde{C}$ with $\tilde{M}$ an L-matrix with respect to rec $\tilde{C}$. If there is a secondary ray in $G_C^{-1}(0)$, then so is in $\tilde{G}_{\tilde{C}}^{-1}(0)$ by

Proposition 8. Therefore, there exists directions $(\Delta\tilde{x}, \Delta\tilde{z}, \Delta\tilde{\lambda}, \Delta\tilde{s}, \Delta t)$ in the reduced space satisfying

$$\begin{aligned} \Delta\tilde{x} - \Delta\tilde{z} &= -\tilde{M}\Delta\tilde{z} + \tilde{r}\Delta t \\ A_{\mathscr{A}\bullet}\Delta\tilde{z} &= 0 \\ A_{\bar{\mathscr{A}}\bullet}\Delta\tilde{z} - \Delta\tilde{s}_{\bar{\mathscr{A}}} &= 0 \\ \Delta\tilde{x} - \Delta\tilde{z} &= -A^T_{\mathscr{A}\bullet}\Delta\tilde{\lambda}_{\mathscr{A}}, \end{aligned} \tag{4}$$

where we have included bound constraints in the matrix A for clarity, and $\mathscr{A}$ and $\bar{\mathscr{A}}$ denote the active and inactive sets, respectively. We then apply Theorem 1 to (4) to get the desired result. □

3.2 Additional processability results

Let us now extend the classes of AVIs that PATHAVI is able to process. The results in Lemmas 2, 3 consider the structure of the whole AVI, not only M and C. As stated in the paragraph following Proposition 7, a 1-manifold generated in $\mathscr{M}_C$ corresponds to another one in $\mathscr{M}_{\tilde{C}}$. Hence, in the following we denote by AVI$(\tilde{C}, \tilde{q}, \tilde{M})$ the AVI corresponding to AVI(C, q, M) with the lineality space projected out as explained in Proposition 7. If M is invertible on lin C, the results can then be applied to the original AVI by noticing that the projections of the directions of the rays on $G_C^{-1}(0)$ are solution to the system of equations (4) in the reduced space.

In Sect. 6.1, we present a friction contact problem where Theorem 2 cannot be applied but the following lemma is appropriate.

Lemma 2 *Consider an AVI*$(\tilde{C}, \tilde{q}, \tilde{M})$ *with* lin $\tilde{C} = \{0\}$. *Suppose that* $\tilde{M}$ *is semimonotone with respect to* rec $\tilde{C}$ *and that for any solution* $z \neq 0$ *of the problem*

$$z \in \operatorname{rec}\ \tilde{C}, \quad \tilde{M}z \in (\operatorname{rec}\ \tilde{C})^D, \quad z^T\tilde{M}z = 0, \tag{5}$$

it holds that

$$z^T\left(\tilde{M}z' + \tilde{q}\right) \geq 0, \quad \forall z' \in \tilde{C}. \tag{6}$$

Then PATHAVI *solves the AVI*$(\tilde{C}, \tilde{q}, \tilde{M})$.

Proof The pivotal method used in PATHAVI fails if an unbounded ray is generated at some iterate (x^k, t^k), $k > 0$. Now suppose that the method generates an unbounded ray. From Lemma 5, we know that $\Delta t = 0$, and $\Delta z \neq 0$ is a solution to (5). This means that for any point x^{k+1} on the ray, we have $\tilde{G}_{\tilde{C}}(x^{k+1}, t^k) = 0$, implying that

$$\left\langle \Delta z, \tilde{G}_{\tilde{C}}(x^{k+1}, t^k)\right\rangle = \langle \Delta z, \tilde{M}z^{k+1} + \tilde{q}\rangle + \langle \Delta z, x^{k+1} - z^{k+1}\rangle + \langle \Delta z, -t^k r\rangle = 0.$$

The first term is non-negative by our assumption, as well as the second one by the normal cone definition. The third one is strictly positive since $-t^k r \in \operatorname{int}(\operatorname{rec}\ \tilde{C})^D$. Hence, we reached a contradiction. □

An additional property on $\tilde{M}$ allows easier checking of condition (6) of Lemma 2.

Corollary 1 *If for any solution z of (5) we have $\langle z', \tilde{M}^T z\rangle \geq 0$, for all $z' \in \tilde{C}$, then the condition (6) reduces to $z^T \tilde{q} \geq 0$ whenever z is a solution to (5).*

We introduce an additional problem class PATHAVI can process.

Lemma 3 *Consider an AVI($\tilde{C}, \tilde{q}, \tilde{M}$) with $\operatorname{lin} \tilde{C} = \{0\}$. Suppose that $\tilde{C}$ is a proper cone, $\tilde{M}$ is copositive with respect to $\tilde{C}$ and that the following implication holds:*

$$z \in \operatorname{rec} \tilde{C}, \quad \tilde{M}z \in (\operatorname{rec} \tilde{C})^D, \quad z^T \tilde{M} z = 0 \quad \Rightarrow \quad z^T \tilde{q} \geq 0. \tag{7}$$

Then the AVI($\tilde{C}, \tilde{q}, \tilde{M}$) has a solution and PATHAVI *finds it.*

Proof Recall from [6, Lemma 4.3], that a copositive matrix is also semimonotone. This implies that $\Delta t = 0$ and that $\Delta z \neq 0$ satisfies the left-hand side of (7). Now let us suppose that at the current iterate x_k, there exists an unbounded ray. Letting $z_{k+1} = z_k + \theta \Delta z$ and computing the inner product $\langle z_{k+1}, \tilde{G}_{\tilde{C}}(x_{k+1}, t_k)\rangle$ yields

$$0 = \left\langle z_{k+1}, \tilde{G}_{\tilde{C}}(x_{k+1}, t_k)\right\rangle = \langle z_{k+1}, \tilde{M} z_{k+1}\rangle + \langle z_{k+1}, \tilde{q}\rangle + \langle z_{k+1}, x_{k+1} - z_{k+1}\rangle \\ + \langle z_{k+1}, -t_k r\rangle.$$

Note that since $\tilde{C}$ is pointed, $\langle z_{k+1}, x_{k+1} - z_{k+1}\rangle \geq 0$ by the definition of the normal cone. The first term is quadratic in θ while the second and third are linear in θ. Therefore, if $\langle \Delta z, \tilde{M} \Delta z\rangle > 0$, then $\langle z_{k+1}, \tilde{G}_{\tilde{C}}(x_{k+1}, t_k)\rangle > 0$ for θ large enough and we reach a contradiction. We are left with the case $\langle \Delta z, \tilde{M} \Delta z\rangle = 0$:

$$0 = \langle z_k, \tilde{q}\rangle - \langle z_k, t_k r\rangle + \langle z_{k+1}, x_{k+1} - z_{k+1}\rangle \\ + \langle z_{k+1}, \tilde{M} z_{k+1}\rangle + \theta(\langle \Delta z, \tilde{q}\rangle + \langle \Delta z, -t_k r\rangle).$$

The sum multiplied by θ is positive since $-t_k r \in \operatorname{int} (\operatorname{rec} \tilde{C})^D$. Now the first two terms are constant and the third and fourth ones are nonnegative. Whence for θ large enough, $\langle z_{k+1}, \tilde{G}_{\tilde{C}}(x_{k+1}, t_k)\rangle$ is positive, which concludes the proof. □

Remark 2 Lemma 3 was already known for the LCP case (that is $\tilde{C} = \mathbb{R}^n_+$): the existence of a solution is given in [7, Theorem 3.8.6]. Here we are able to provide a constructive proof for an AVI($\tilde{C}, \tilde{q}, \tilde{M}$) over a proper cone.

Let us present an AVI(C, q, M) that satisfies the conditions of Lemma 3 where M is not an L-matrix. Suppose that $C \subseteq \mathbb{R}^{n+1}_+$ is a polyhedral solid cone, $M = \begin{pmatrix} I_n & 0 \\ \mathbf{1}_n^T & 0 \end{pmatrix}$, with $\mathbf{1}_n$ the vector of ones of size n and $q = (0_n, 1)^T$. The solution set of the system $x \in C, Mx = 0$ and $x^T Mx = 0$ is $\{(0_n, \alpha)^T, \alpha \geq 0\}$. Note that if $x = (0_n, \alpha)^T, \alpha > 0$, then $Mx = 0$ and $x^T Mx = 0$. However, for any nonzero vector $x' = (x_1'^T, \alpha')^T$ in C, $-M^T x' = (-I_n x_1'^T - \alpha' \mathbf{1}^T, 0)^T \notin C^D$. Therefore, condition (b) of the L-matrix fails to hold. On the other hand, we can readily check that M is copositive with respect to C and that for any $x = (0_n, \alpha)^T, \alpha \geq 0, x^T q = \alpha \geq 0$, so that Lemma 3 can be used.

4 Computing an implicit extreme point for a ray start

In this section, we describe how to compute an implicit extreme point satisfying the sufficient conditions for a ray start and the complementary basis associated with it so that we can start complementary pivoting at that implicit extreme point. The overall procedure is as follows: (i) we first compute an initial basic feasible solution using a linear programming (LP) solver, i.e., CPLEX or GUROBI; (ii) as the initial solution might not be an implicit extreme point, we may perform additional pivoting to move to an implicit extreme point; (iii) using the basis information associated with the implicit extreme point, we then construct a complementary system of equations such that a unique solution to that system of equations is an implicit extreme point satisfying the sufficient conditions for a ray start. The use of the existing LP solver, which has a fast sparse linear algebra engine and pivoting method, as well as the use of sparse linear algebra engine for complementary pivoting enables PATHAVI to fully exploit the sparse representation of the given AVI. This makes our method efficient for large-scale AVI problems as illustrated by the examples in Sect. 6.2. More details on the overall computational procedure are given in the Appendix as Algorithm 2.

We start with an introduction to some terminology and notational conventions for describing a basic solution of an LP problem. We follow notation used in [4]. Suppose that we run an LP solver over the following problem: minimize $c^T z$ subject to $Az - b \in K$ and $l \leq z \leq u$. Without loss of generality, we assume that we have eliminated all fixed variables. For each solution z obtained from the LP solver, we have four index sets, B, N_l, N_u, and N_{fr}, for variables and two index sets, $\mathscr{A}$ and $\bar{\mathscr{A}}$, for constraints described by A and b[1]. Table 1 lists the properties of the index sets and the solution z. In Table 1, if $l_B \leq z_B \leq u_B$, we say that z is a basic feasible solution. Otherwise, we say that z is a basic solution. Note that we have $|\mathscr{A}| = |B|$ in Table 1 as the basis matrix $\mathbf{B}$ is invertible. Hence, the submatrix $A_{\mathscr{A}B}$ of $\mathbf{B}$ is square and invertible.

We first describe how to compute an implicit extreme point of C. For a given AVI(C, q, M), we formulate and solve the following LP problem using an LP solver:

$$\begin{aligned} \text{minimize} \quad & 0^T z, \\ \text{subject to} \quad & Az - b \in K, \\ & l \leq z \leq u. \end{aligned} \tag{LP}$$

We put zero objective coefficients in the (LP) so that the (LP) returns whenever it finds a basic feasible solution. If we have an intuition about where to start complementary pivoting, then we could try to solve the (LP) with different objective coefficients.

Assuming that the (LP) is feasible, a basic feasible solution z^0 from the LP solver with the corresponding index sets is an extreme point if $N_{fr} = \emptyset$. When $N_{fr} \neq \emptyset$, z^0 might not be an implicit extreme point. In this case, we move from z^0 to an implicit extreme point by doing additional pivoting in a way that forces as many nonbasic free variables to become basic variables. Algorithm 1 in the Appendix describes the piv-

[1] These index sets can be obtained using `CPXgetbase()` for CPLEX, for example.

Table 1 Index sets and a basis matrix describing a basic solution z of an LP problem. Assume that $z \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$

$B \cup N_l \cup N_u \cup N_{fr} = \{1, \ldots, n\}$ and $B, N_l, N_u,$ and N_{fr} are mutually exclusive.
$B :=$ a set of basic variables indices
$N_l :=$ a set of nonbasic variables indices at their finite lower bounds
$N_u :=$ a set of nonbasic variables indices at their finite upper bounds
$N_{fr} :=$ a set of nonbasic free variables indices
$\mathscr{A} \cup \bar{\mathscr{A}} = \{1, \ldots, m\}$ with $\mathscr{A} \cap \bar{\mathscr{A}} = \varnothing$
$\mathscr{A} :=$ a set of active constraints indices, i.e., $A_{\mathscr{A}\bullet} z = b_{\mathscr{A}}$
$\bar{\mathscr{A}} :=$ a set of inactive constraints indices
$\mathbf{B} = \begin{bmatrix} A_{\mathscr{A}B} & 0 \\ A_{\bar{\mathscr{A}}B} & \pm I_{\bar{\mathscr{A}}} \end{bmatrix}$ is an invertible basis matrix where $I_{\bar{\mathscr{A}}}$ is an identity matrix of size $\lvert\bar{\mathscr{A}}\rvert \times \lvert\bar{\mathscr{A}}\rvert$
$z_B = A^{-1}_{\mathscr{A}B}\left(b_{\mathscr{A}} - A_{\mathscr{A}N} z_N\right),\ z_{N_l} = l_{N_l},\ z_{N_u} = u_{N_u},\ z_{N_{fr}} = 0,\ N = N_l \cup N_u \cup N_{fr}$

oting procedure. After applying Algorithm 1, for each $j \in N_{fr}$ and $d^j = A^{-1}_{\mathscr{A}B} A_{\mathscr{A},j}$ if there exists k such that $d^j_k \neq 0$, then the basic variable corresponding to the kth position in B is a free variable. Otherwise, the variable z_j must have been pivoted in by Algorithm 1. Also, note that Algorithm 1 doesn't change the properties described in Table 1. Using Algorithm 1, we obtain the following result.

Proposition 9 *Suppose that we have applied Algorithm 1. Then the new point, denoted by $\bar{z}^0$, constructed from z^0 through Algorithm 1 is an implicit extreme point of C. We have* $\dim(\operatorname{lin} C) = |N_{fr}|$ *and the following set of vectors is a basis for the lineality space of C:*

$$\bigcup_{j \in N_{fr}} \{v^j\}, \quad v^j_k = \begin{cases} (A^{-1}_{\mathscr{A}B} A_{\mathscr{A},j})_k & \text{if } k \in B, \\ 0 & \text{if } k \in N_l \cup N_u, \\ 0 & \text{if } k \in N_{fr}, k \neq j, \\ 1 & \text{if } k = j. \end{cases}$$

Proof Clearly, $\bar{z}^0 \in C$ as we do a ratio test to move the point. We first show that $\operatorname{lin} C = |N_{fr}|$ and $\{v^j\}_{j \in N_{fr}}$ is a basis for the lineality space of C. For each $j \in N_{fr}$, if $v^j_k \neq 0$ for $k \in B$, then we have $l_k = -\infty$ and $u_k = \infty$ as discussed in the previous paragraph. It follows that $\bar{z}^0 + \lambda v^j \in C$ for all $\lambda \in \mathbb{R}$. By [27, Theorem 8.3], $v^j \in \operatorname{rec} C \cap (-\operatorname{rec} C)$. Thus $v^j \in \operatorname{lin} C$. By construction of v^j, we see that v^j's are linearly independent. This implies that $\dim(\operatorname{lin} C) \geq |N_{fr}|$. As $\dim(N_C(\bar{z}^0)) \geq |B| + |N_l| + |N_u|$ and $N_C(\bar{z}^0) \subset (\operatorname{lin} C)^{\perp}$ as shown in Proposition 6(d), it follows that $\dim(\operatorname{lin} C) = |N_{fr}|$ and $\{v^j\}_{j \in N_{fr}}$ is a basis for the lineality space of C.

We now prove that $\bar{z}^0$ is an implicit extreme point of C. Suppose that $\bar{z}^0 = \lambda z^1 + (1-\lambda) z^2$ for some $z^1, z^2 \in C$ and $\lambda \in (0, 1)$. Define $d^k = \sum_{j \in N_{fr}} (-z^k_j v^j)$ and set $\tilde{z}^k = z^k + d^k$ for $k = 1, 2$. We then have $\tilde{z}^k_j = 0$ for $j \in N_{fr}$ and $\tilde{z}^k \in C$ as $d^k \in \operatorname{lin} C$ for $k = 1, 2$. As $\bar{z}^0 = \lambda z^1 + (1-\lambda) z^2$, $\bar{z}^0 = \lambda \tilde{z}^1 + (1-\lambda)\tilde{z}^2 - (\lambda d^1 + (1-\lambda) d^2)$. We have $\lambda d^1+(1-\lambda)d^2 = \sum_{j\in N_{fr}} \left(-(\lambda z_j^1 + (1-\lambda)z_j^2)v^j\right)$. As $\bar{z}^0_{N_{fr}} = \bar{z}^1_{N_{fr}} = \bar{z}^2_{N_{fr}} = 0$, $v_j^j = 1$, and $v_h^j = 0$ for $h \in N_{fr}$, $h \neq j$, we see that $\lambda d^1+(1-\lambda)d^2 = 0$. Therefore, $\bar{z}^0 = \lambda\bar{z}^1 + (1-\lambda)\bar{z}^2$. It follows that $\bar{z}^0 = \bar{z}^1 = \bar{z}^2$. Thus, $\bar{z}^0 - z^k = d^k \in \operatorname{lin} C$ for $k = 1, 2$, which implies that $\bar{z}^0$ is an implicit extreme point of C. □

Using the implicit extreme point $\bar{z}^0$ of C and the index sets $(B, N_l, N_u, N_{fr}, \mathscr{A}, \bar{\mathscr{A}})$ associated with it, we construct an initial complementary basis and compute an implicit extreme point satisfying the sufficient conditions for a ray start from that complementary basis. To prove the invertibility of our initial complementary basis, we first introduce the following technical result derived from [21, Lemma 3.6].

Corollary 2 *Suppose that we have index sets $(B, N_l, N_u, N_{fr}, \mathscr{A}, \bar{\mathscr{A}})$ associated with an AVI(C, q, M) with a nonempty N_{fr}. Then Z is invertible if and only if $\tilde{W}^T\tilde{M}\tilde{W}$ is invertible, where*

$$Z = \begin{bmatrix} M_{BB} & M_{BN_{fr}} & -A^T_{\mathscr{A}B} \\ M_{N_{fr}B} & M_{N_{fr}N_{fr}} & -A^T_{\mathscr{A}N_{fr}} \\ A_{\mathscr{A}B} & A_{\mathscr{A}N_{fr}} & 0_{\mathscr{A}} \end{bmatrix}, \quad \tilde{M} = \begin{bmatrix} M_{BB} & M_{BN_{fr}} \\ M_{N_{fr}B} & M_{N_{fr}N_{fr}} \end{bmatrix},$$

$$\tilde{W} = \begin{bmatrix} -A^{-1}_{\mathscr{A}B}A_{\mathscr{A}N_{fr}} \\ I_{N_{fr}} \end{bmatrix}.$$

Proof As $A_{\mathscr{A}B}$ is square and invertible, $\ker\left[A_{\mathscr{A}B}\ A_{\mathscr{A}N_{fr}}\right] = \operatorname{im} \tilde{W}$. The result follows from [21, Lemma 3.6]. □

We are now ready to present our initial complementary basis and an implicit extreme point satisfying the sufficient conditions for a ray start.

Proposition 10 *For a given AVI(C, q, M), suppose that we have an implicit extreme point $\bar{z}^0$ and the index sets $(B, N_l, N_u, N_{fr}, \mathscr{A}, \bar{\mathscr{A}})$ associated with $\bar{z}^0$. Then the matrix on the left-hand side of the following system of equations is invertible if and only if M is invertible on the lineality space of C. Also $z = (z_B, z_{N_{fr}}, \bar{z}^0_{N_l}, \bar{z}^0_{N_u})$ in a solution to the system of equations satisfies $z \in \bar{z}^0 + \operatorname{lin} C$, i.e., z is an implicit extreme point of C by Proposition 2 in Sect. 3, and $Mz + q \in \operatorname{aff}(N_C(z))$.*

$$\begin{bmatrix} M_{BB} & M_{BN_{fr}} & -A^T_{\mathscr{A}B} & 0 & 0 & 0 \\ M_{N_lB} & M_{N_lN_{fr}} & -A^T_{\mathscr{A}N_l} & -I_{N_l} & 0 & 0 \\ M_{N_uB} & M_{N_uN_{fr}} & -A^T_{\mathscr{A}N_u} & 0 & I_{N_u} & 0 \\ M_{N_{fr}B} & M_{N_{fr}N_{fr}} & -A^T_{\mathscr{A}N_{fr}} & 0 & 0 & 0 \\ A_{\mathscr{A}B} & A_{\mathscr{A}N_{fr}} & 0 & 0 & 0 & 0 \\ A_{\bar{\mathscr{A}}B} & A_{\bar{\mathscr{A}}N_{fr}} & 0 & 0 & 0 & -I_{\bar{\mathscr{A}}} \end{bmatrix} \begin{bmatrix} z_B \\ z_{N_{fr}} \\ \lambda_{\mathscr{A}} \\ w_{N_l} \\ v_{N_u} \\ s_{\bar{\mathscr{A}}} \end{bmatrix} = \begin{bmatrix} -q_B - M_{BN}\bar{z}^0_N \\ -q_{N_l} - M_{N_lN}\bar{z}^0_N \\ -q_{N_u} - M_{N_uN}\bar{z}^0_N \\ -q_{N_{fr}} - M_{N_{fr}N}\bar{z}^0_N \\ b_{\mathscr{A}} - A_{\mathscr{A}N}\bar{z}^0_N \\ b_{\bar{\mathscr{A}}} - A_{\bar{\mathscr{A}}N}\bar{z}^0_N \end{bmatrix}.$$

Proof The matrix on the left-hand side of the system of equations is invertible if and only if the matrix Z defined in Corollary 8 is invertible. This is because of the identity submatrices of it, $-I_{N_l}$, I_{N_u}, and $-I_{\bar{\mathscr{A}}}$. The columns of the matrix $W =$

$(-A_{\mathscr{A}B}^{-1}A_{\mathscr{A}N_{fr}} \quad I_{N_{fr}} \quad 0_{N_l} \quad 0_{N_u})^T$ form a basis of the lineality space of C. Note that $W^TMW = \tilde{W}^T\tilde{M}\tilde{W}$ where $\tilde{W}$ and $\tilde{M}$ are the matrices defined in Corollary 8. Therefore, the matrix is invertible if and only if M is invertible on the lineality space of C.

We now show that the z constructed from the solution to the linear system satisfies $z \in \bar{z}^0 + \text{lin } C$. The fifth equation gives us

$$z_B = -A_{\mathscr{A}B}^{-1}A_{\mathscr{A}N_{fr}}z_{N_{fr}} + A_{\mathscr{A}B}^{-1}\left(b_{\mathscr{A}} - A_{\mathscr{A}N}\bar{z}_N^0\right).$$

If $z_{N_{fr}} = 0$, then $z_B = \bar{z}_B^0$ and $z = \bar{z}^0$. For $z_{N_{fr}} \neq 0$, we have $z = \bar{z}^0 + Wz_{N_{fr}}$. As W is a basis for the lineality space of C, it follows that $z \in \bar{z}^0 + \text{lin } C$. Since $\bar{z}^0$ is an implicit extreme point, z enjoys the same property by Proposition 2.

Finally, from the first four equations of the given system, we see that $Mz + q \in \text{aff }(N_C(z))$. □

5 Worst-case performance comparison: AVI versus MCP reformulation

In this section, we introduce the MCP reformulation of an AVI and analyze worst-case performance of the two formulations in Sects. 5.1 and 5.2, respectively. We assume that both problems are solved using the same complementary pivoting method. Computational results comparing the two formulations are presented in Sect. 6, and demonstrate the effectiveness of working on the original manifold $\mathscr{M}_C$ (see Tables 2–4; Fig. 3).

5.1 MCP reformulation

A linear MCP is defined as follows: for an affine function $F(z) = Mz + q$ and a box constraint $B_1 := \prod_{j=1}^n [l_j, u_j]$, z is a solution to the MCP(B_1, q, M) if $Mz+q = w-v$ with $z \in B_1$, $w, v \in \mathbb{R}_+^n$, $(z-l)^Tw = 0$, and $(u-z)^Tv = 0$.

It is well known [10, 4] that an AVI(C, q, M) can be reformulated as an MCP$(B_1 \times B_2, \tilde{q}, \tilde{M})$, where

$$\begin{aligned} &B_1 = \prod_{j=1}^n [l_j, u_j], \quad B_2 = \{\lambda \in \mathbb{R}^m \mid \lambda \in K^D\}, \\ &\tilde{M} = \begin{bmatrix} M & -A^T \\ A & 0 \end{bmatrix}, \quad \tilde{q} = \begin{bmatrix} q \\ -b \end{bmatrix}. \end{aligned} \qquad \text{(MCP-reform)}$$

By Facchinei and Pang [15, Proposition 1.2.1], z^* is a solution to the AVI(C, q, M) if and only if there exists λ^* such that (z^*, λ^*) is a solution to the MCP$(B_1 \times B_2, \tilde{q}, \tilde{M})$. Therefore, we can solve an AVI(C, q, M) by solving its MCP$(B_1 \times B_2, \tilde{q}, \tilde{M})$ reformulation and vice versa. The solver PATH [9], one of the most efficient MCP solvers, uses this MCP reformulation when it processes an AVI.

Although the two formulations are equivalent, they do not share the same theoretical properties. This is mainly because they look at different feasible regions and recession cones, which also results in different PL manifolds on which the complementary pivoting is performed. For the MCP($B_1 \times B_2, \tilde{q}, \tilde{M}$) reformulation, a PL $(n+m+1)$-manifold $\mathscr{M}_{B_1 \times B_2}$ is built where the full-dimensional cells are defined by the nonempty faces and the normal cones of the set $B_1 \times B_2$, which doesn't consider the polyhedral constraints $Az - b \in K$. For the AVI(C, q, M) formulation, a PL $(n+1)$-manifold $\mathscr{M}_C$ is constructed based on the nonempty faces and normal cones of C, which includes the polyhedral constraints $Az - b \in K$ explicitly.

5.2 Worst-case performance analysis

In the worst case, the complementary pivoting method ends up going through all the full-dimensional cells of the underlying PL manifold. As each iteration of the complementary pivoting method corresponds to the traversal of one full-dimensional cell assuming nondegeneracy or lexicographic pivoting, the maximum number of iterations is the total number of the full-dimensional cells, which is finite but could be exponential in the number of constraints. Therefore, we compare worst-case performance of the two formulations by counting the number of the full-dimensional cells of the PL manifold that each formulation generates.

By construction, the number of the full-dimensional cells is equivalent to the number of the nonempty faces of the polyhedral convex set being considered [25, 6]. Thus, we count the number of the nonempty faces of both $B_1 \times B_2$ and C.

Let NNF(S) denote the number of the nonempty faces of a polyhedral convex set S. To count the number of the nonempty faces, we start with building blocks defining a polyhedral convex set: intervals $[l, u]$ in $\mathbb{R}$ and linear constraints $a^T z - b \in K$. For a closed interval $[l, u]$ in $\mathbb{R}$, the number of the nonempty faces is as follows:

$$\mathrm{NNF}([l,u]) = \begin{cases} 1 & \text{if } -\infty = l < u = \infty \text{ or } -\infty < l = u < \infty, \\ 2 & \text{if } -\infty = l < u < \infty \text{ or } -\infty < l < u = \infty, \\ 3 & \text{if } -\infty < l < u < \infty. \end{cases} \tag{8}$$

For a halfspace or a hyperplane defined by a linear constraint $a^T z - b \in K$ where $a \neq 0$ and $b \in \mathbb{R}$, the number of the nonempty faces is as follows:

$$\mathrm{NNF}(\{z \in \mathbb{R}^n \mid a^T z - b \in K\}) = \begin{cases} 2 & \text{if } K = \mathbb{R}_+ \text{ or } K = \mathbb{R}_-, \\ 1 & \text{if } K = \{0\}. \end{cases} \tag{9}$$

Based on (8) and (9), we can compute an upper bound on the number of the nonempty faces of a polyhedral convex set.

Lemma 4 *Let C be a polyhedral convex set defined by $C = \{z \in \mathbb{R}^n \mid Az - b \in K, l \le z \le u\}$. Then*

$$\mathrm{NNF}(C) \le \prod_{j=1}^{n} \mathrm{NNF}([l_j, u_j]) \times \prod_{i=1}^{m} \mathrm{NNF}\left(\{z \in \mathbb{R}^n \mid A_{i\bullet}^T z - b_i \in K_i\}\right), \tag{10}$$

where the symbol $\prod_j$ denotes multiplication over indexed terms in this case.

Proof Let $C_j = \{z \in \mathbb{R}^n \mid z_j \in [l_j, u_j]\}$ for $j = 1, \ldots, n$ and $C_{n+i} = \{z \in \mathbb{R}^n \mid A_{i,:}^T z - b_i \in K_i\}$ for $i = 1, \ldots, m$. Then $C = \cap_{i=1}^{n+m} C_i$. By [26, Corollary 4.2.15], F is a face of C if and only if $F = \cap_{i=1}^{n+m} F_i$ where F_i is a face of C_i for $i = 1, \ldots, n+m$. The result follows. $\square$

In Lemma 4, there could be a large gap between NNF(C) and its upper bound. The upper bound counts all the possible combinations of the faces of each constraint regardless of their feasibility. When C has only box constraints, i.e., $C = \{z \in \mathbb{R}^n \mid l \leq z \leq u\}$, then equality holds in (10). But, in other cases, the upper bound could be much larger than NNF(C) as not every combination corresponds to a nonempty face of C. For example, if $C = \{z \in \mathbb{R}^2 \mid z_1 + z_2 \geq -1, -z_1 + z_2 \geq -1, z_1 - z_2 \geq -1, -z_1 - z_2 \geq -1, -1 \leq z_1, z_2 \leq 1\}$, we have NNF($C$) $= 9$. However, the upper bound is 144. It turns out that there are many infeasible combinations, i.e., all the combinations having $z_1 = -1$ and $z_2 = 1$.

Using Lemma 4, we prove that the maximum number of cells for the AVI(C, q, M) manifold is smaller or equal to the cells in the MCP($B_1 \times B_2, \tilde{q}, \tilde{M}$) manifold.

Proposition 11 *Let an AVI(C, q, M) formulation and its MCP($B_1 \times B_2, \tilde{q}, \tilde{M}$) reformulation defined in* (MCP-reform) *be given. Then the number of the full-dimensional cells of the* PL *($n + 1$)-manifold $\mathscr{M}_C$ is less than or equal to the number of the full-dimensional cells of the* PL *($n + m + 1$)-manifold $\mathscr{M}_{B_1 \times B_2}$.*

Proof By [26, Proposition 4.2.12], NNF($B_1 \times B_2$) $=$ NNF(B_1)$\times$NNF(B_2). By applying the same proposition, we have NNF(B_1) $= \prod_{j=1}^n$ NNF($[l_j, u_j]$) and NNF(B_2) $= \prod_{i=1}^m$ NNF($[l_i^\lambda, u_i^\lambda]$) where l_i^λ and u_i^λ are lower and upper bounds on λ_i variable. Using (8) and (9), we see that $\prod_{i=1}^m$ NNF($\{z \in \mathbb{R}^n \mid A_{i\bullet}^T z - b_i \in K_i\}$) $= \Pi_{i=1}^m$NNF($[l_i^\lambda, u_i^\lambda]$). By Lemma 4, the result follows. $\square$

Based on Proposition 11, we expect that PathAVI will take fewer iterations than Path, which solves the MCP reformulation, since in the worst case both may visit every cell in the manifold. This is confirmed by the computational results in Sects. 6.3–6.5.

6 Computational results

In this section, we present computational results of PathAVI highlighting its computational benefits of preserving the problem structure and its robustness and efficiency compared to Path version 4.7 [9,17], an established solver for AVIs which uses the MCP reformulation. The majority of the examples are based on friction contact models, which we briefly describe in Sect. 6.1. Section 6.2 shows improved performance of PathAVI using the original AVI formulation containing nontrivial lineality space over its equivalent reduced form that does not contain lines. Sections 6.3–6.5 compare performance of PathAVI and Path over friction contact problems, compact sets, and Nash equilibrium problems, respectively and demonstrate the advantages of the stronger theory associated with PathAVI.

All experiments were performed on a Linux machine with Intel Xeon(R) E7-4850 2.00 GHz processor and 256 GB of memory. PATH and PATHAVI were compiled was compiled using GNU gcc version 4.4.7 and its interfaces were linked to GAMS. All problem instances were written in GAMS using the EMP syntax for variational inequalities [16]. We set the time limit to 1 hour and major/minor iteration limits to 20 and 10^5, respectively.

6.1 Friction contact problem

Coulomb or dry friction is a ubiquitous phenomenon when mechanical systems interact via contact with each other. Consider a mechanical system with n_{dof} degrees of freedom, n_d bodies and n_c contacts. The number of degrees of freedom depends on the type of system we consider, i.e., if we have rigid bodies, $n_{\text{dof}} = 6n_d$. However, if we have deformable bodies, then this number is typically larger and depends on the modeling used for the bodies. For each two bodies in contact at a single point, we denote by $u^{(k)} := (u_n^{(k)}, u_t^{(k)})^T \in \mathbb{R}_+ \times \mathbb{R}^2$ the relative (or local) velocity between them and the reaction force is given by $r^{(k)} = (r_n^{(k)}, r_t^{(k)})$. One of the numerous ways (see [2] for a list of them) to model the dynamics of a system with Coulomb friction is:

$$\begin{aligned} -u_n^{(k)} &\in N_{\mathbb{R}_+}(r_n^{(k)}) & k = 1, \ldots, n_c \quad \text{and} \quad & Mv = Hr + f \\ -u_t^{(k)} &\in N_{r_n^{(k)}\mu^{(k)}D}(r_t^{(k)}) & & u = H^T v + w, \end{aligned} \tag{11}$$

with $M \in \mathbb{R}^{n_{\text{dof}} \times n_{\text{dof}}}$, $H \in \mathbb{R}^{n_{\text{dof}} \times 3n_c}$, $\mathbb{R}^{3n_c} \ni r := [r_n^{(1)}, r_t^{(1)}, \ldots, r_n^{(n_c)}, r_t^{(n_c)}]^T$ and $\mathbb{R}^{3n_c} \ni u := [u_n^{(1)}, u_t^{(1)}, \ldots, u_n^{(n_c)}, u_t^{(n_c)}]^T$, see Fig. 1 for an example. It is shown in [18] that (11) is equivalent to the following complementarity problem over a second order cone:

$$0 \in \begin{pmatrix} M & -H & 0 \\ H^T & 0 & E \\ \bar{H}^T & 0 & E \end{pmatrix} \begin{pmatrix} v \\ r \\ y \end{pmatrix} + \begin{pmatrix} -f \\ w \\ \bar{w} \end{pmatrix} + N_X \begin{pmatrix} v \\ r \\ y \end{pmatrix} \qquad X := \mathbb{R}^{n_{\text{dof}}} \times K \times K, \tag{12}$$

where $K := \prod_{k=1}^{n_c} K_{\mu^k}$ and $K_{\mu^k} := \{(t, tx) \mid t \in \mathbb{R}_+, x \in \mu^k D\}$, D being the unit disk in $\mathbb{R}^2$. If we split H as $[H_{1,n}, H_{1,t}, \ldots, H_{i,n}, H_{i,t}, \ldots, H_{n_c,n}, H_{n_c,t}]$ with $H_{k,n} \in \mathbb{R}^{n_{\text{dof}}}$, $H_{k,t} \in \mathbb{R}^{n_{\text{dof}} \times 2}$, then $\bar{H} := [0_n, H_{1,t}, \ldots, 0_n, H_{k,t}, \ldots, 0_n, H_{n_c,t}]$. Similarly, letting $(w_{k,n}, w_{k,t}) \in \mathbb{R} \times \mathbb{R}^2$ we have $w = [w_{1,n}, w_{1,t}, \ldots, w_{n_c,n}, w_{n_c,t}]$ and $\bar{w} := [0, w_{1,t}, \ldots, 0, w_{n_c,t}]$. Finally, the diagonal matrix $E \in \mathbb{R}^{3n_c \times 3n_c}$ is based on the vector $(1, 0, 0)^T$ repeated n_c times. The variable $y := [y_{1,n}, y_{1,t}, \ldots, y_{n_c,n}, y_{n_c,t}]^T$ with $(y_{k,n}, y_{k,t}) \in \mathbb{R} \times \mathbb{R}^2$, is introduced to ensure that the modified local velocity $u + Ey$ belongs to K^D. Since the cone K is not polyhedral, we need to approximate K to get an AVI from (12). Then, we have to solve a sequence of AVIs until one of the solutions also satisfies (12) up to the specified tolerance. Computationally, the most demanding step is the solution of the first AVI in the sequence. Furthermore, we focus here on the case where it makes sense to perform a ray start. Hence, we solve the AVI

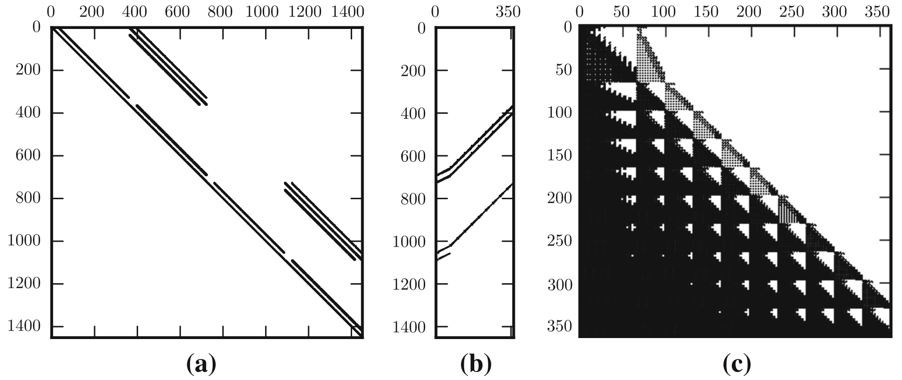

Fig. 1 Nonzero patterns of the matrices M (size: 1452×1452, nnz: 11, 330), H (size: 1452×363, nnz: 1747) and $W := H^T M^{-1} H$ (size: 363×363, nnz: 56, 770)

that would correspond to the first iteration and with an "anisotropic" approximation of K. For each contact we construct a finitely representable polyhedral approximation D_p^k of the disk $\mu^k D$. Then, the cone K is approximated by $K_p := \prod_k K_p^k$, with $K_p^k := \{(t, tx) \mid t \in \mathbb{R}_+, x \in D_p^k\}$. Finally, with a slight abuse of notation, we redefine $X := \mathbb{R}^{n_{\text{dof}}} \times K_p \times K_p$ and refer to (12) as an AVI. It can be verified that PATHAVI processes the AVI (12) if $w \in (\ker H \cap K)^D$ by applying Lemma 2. It is noteworthy that this condition is exactly the one given in [19] for the existence of solution to the complementarity problem over a second order cone (12). If we solely rely on the L-matrix property, we need to assume that $\ker H = \{0\}$, which fails in many instances, for example when a 4-legged chair is in contact with flat ground.

6.2 Computational benefits of preserving the problem structure

The problem data (M, H, f, w) for the following numerical results were obtained from simulations of deformable bodies with the LMGC90 [12] software and using a solver from SICONOS [3]. In the following, we focus on a simple example where two deformable cubes are on top of another. During the simulation, the number of contacts varies between 80 and 120. The shape of M and H is given in Fig. 1.

It is noteworthy that if we have to remove the lineality space, that is to compute W, then the sparse structure of the problem is destroyed (see Fig. 1c): the number of nonzero elements is increased by a factor of five. It is expected that the linear algebra computations will be more expensive in the reduced space formulation than in the original one because of this large increase of nonzero entries. This has been verified on instances that have the same kind of structure as the matrices depicted in Fig. 1.

Both problems are AVIs, but as shown on Fig. 2, PATHAVI working in the original space is always faster and most of the time is at least twice as fast as PATHAVI working in the reduced space. The time in the reduced space does not even take into account the transformation of the problem data, that is the computation of the W matrix.

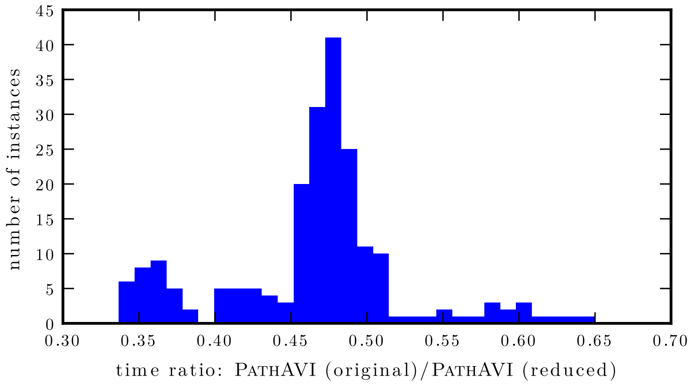

Fig. 2 Comparison in terms of speed between the resolution in the original space and the reduced one. The number of iteration was the same for all the 209 instances

6.3 Multibody friction contact problems

When the bodies are rigid, it is common in the contact mechanic community to eliminate the velocity v. The problem is formulated in a reduced space $K_p \times K_p$ (defined in Sect. 6.1) and the AVI is

$$0 \in \begin{pmatrix} W & E \\ \bar{W} & E \end{pmatrix} \begin{pmatrix} r \\ y \end{pmatrix} + \begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix} + N_{K_p \times K_p} \begin{pmatrix} r \\ y \end{pmatrix}, \tag{13}$$

where $W := H^T M^{-1} H$ and $\bar{W} := \bar{H}^T M^{-1} H$, $\omega := w + H^T M^{-1} f$ and $\bar{\omega} := \bar{w} + \bar{H}^T M^{-1} f$. The lineality space is then trivial in this formulation.

We present computational results using the problem data (W, μ and q) from the FCLIB collection[2] [1], which aims at providing challenging instances of the friction contact problem. Let us highlight a few facts based on the data presented in Table 2:

- PATHAVI can solve all the instances with the linear algebra package UMFPACK [8] ("pathavi/UMFPACK") and is generally faster than PATH.
- Some problems are numerically challenging and the behavior of the solver changes with the linear algebra routines. Specifically, on those problems using LUSOL ("pathavi/default") leads to 20 failures. That can be reduced by using the block-LU updates [14] ("pathavi/LUSOL-blu"). These errors are caused by some numerical issues in the linear algebra package. This illustrates the importance of being able to change the linear algebra engine in PATHAVI.
- PATH is unable to perform a ray start in many instances (whenever ker W is not trivial); in these cases, PATHAVI significantly outperforms PATH (with or without the crash method).

Let us explain the failure types: "Solver error" means that the first basis matrix could not be factorized, despite the use of artificial variables to overcome the rank

[2] The collection of problem can be freely downloaded by visiting http://fclib.gforge.inria.fr.

Table 2 Statistics for 4579 friction contact problems of the form (13)

Solver/profile	# Failed	Failure type			
		Solver error	Stalled	Time	Iteration
pathavi/UMFPACK	0	0	0	0	0
pathavi/default	20	0	0	0	20
pathavi/LUSOL-blu	3	0	0	0	3
path/default	2060	535	1525	0	0
path/no crash	108	99	0	8	1

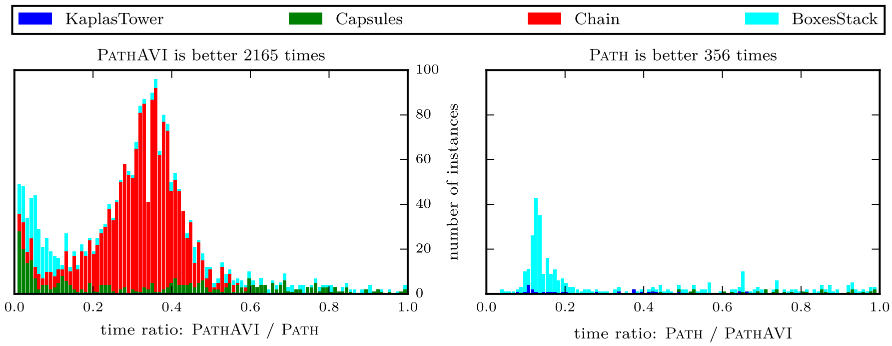

Fig. 3 Time comparison between PATH and PATHAVI (color figure online)

deficiency. "Stalled" means that a solver tried various strategies but failed to find a solution at the requested accuracy and consequently gave up. Note that this never occurred with PATHAVI on this set of problems. "Time" (or "Iteration") signals that the time (or iteration) limit has been reached. The convergence tolerance is set to a low value: $\sqrt{N} \times 10^{-9}$, where N is the number of contacts. This value is lower than the default tolerance of PATH (that is already considered quite demanding).

The default behavior of PATH ("path/default") leads to many failures: the crash method is inappropriate for such models. However, even without the crash procedure ("path/no crash"), PATH still fails at a higher rate than PATHAVI.

We further compare PATH and PATHAVI on their default settings on the subset of problems solved by both. The results are presented in Fig. 3 in terms of time ratios. First note that PATHAVI is faster than PATH in the majority of cases, and that it usually finds a solution in less than half the time of PATH. The spike on the right plot, when PATH finds the solution faster than PATHAVI, is explained by the fact that the crash procedure in PATH performed well in those instances ($<$10% of the examples).

6.4 AVIs over compact sets

One strong implication of Theorem 2 is that when C is compact (so that rec $C = \{0\}$) PATHAVI can process an AVI(C, q, M) with arbitrary M and q. In contrast, this does

Table 3 Performance of PATHAVI and PATH over compact sets

Name	(#constrs, #vars)	[nnz(A), nnz(M)]	Number of iterations		Elapsed time (s)	
			PATHAVI	PATH	PATHAVI	PATH
CVXQP1_M	(500, 1000)	(2495, 999)	3119	Fail	0.459	Fail
CVXQP2_M	(250, 1000)	(1746, 999)	33,835	Fail	2.827	Fail
CVXQP3_M	(750, 1000)	(3244, 999)	360	3603	0.105	1.992
CONT-050	(2401, 2597)	(14,597, 6407)	11	382	2.753	272.429
CONT-100	(9801, 10,197)	(59,197, 98,875)	3	Fail	174.267	Fail

not hold for the MCP reformulation as the underlying feasible region $[B_1 \times B_2]$ of it may not be compact although C is compact. This is because whenever the AVI contains polyhedral constraints the associated λ variables in the MCP reformulation are only constrained to lie in the unbounded set B_2. We construct five AVI instances by taking compact feasible regions from [22] having finite lower and upper bounds and by randomly generating M and q such that the resultant AVI has an M with negative eigenvalues.

Table 3 presents some computational results. As expected, PATHAVI is able to solve all the instances, whereas PATH fails to solve three of them. Also, on the two problem instances where both solvers are able to solve, PATHAVI shows 10–30 times fewer iterations, and a similarly decreased elapsed time. These properties hold for a wide selection of instances and the above table is just provided for expository purposes.

6.5 Nash equilibrium problems

Another application of AVIs is to Nash equilibrium problems. In a Nash equilibrium problem, there are multiple agents each of which minimizing its own objective function, and each agent's objective function not only depends on the agent's decision but also other agents' decisions. For example, a typical Nash equilibrium problem computes a solution satisfying

$$x_i^* \in \operatorname*{arg\,min}_{x_i \in X_i} h_i\left(x_i, x_{-i}^*\right), \quad \text{for} \quad i = 1, \ldots, N, \tag{NEP}$$

where we note that each ith agent's objective function h_i takes its own decision, denoted by x_i, and other agents' decisions, denoted by x_{-i}.

We generated six instances of Nash equilibrium problems, where each X_i is a polyhedral convex set and h_i is continuously differentiable in x and convex quadratic in x_i for each fixed x_{-i}. Specifically, h_i takes the following form:

$$h_i(x_i, x_{-i}) = \frac{1}{2} x_i^T Q_i x_i + x_i^T Q_{-i} x_{-i} + c_i^T x_i + d_i^T x_{-i},$$

where Q_i is symmetric positive definite.

Table 4 Performance of PATHAVI and PATH over the NEPs

Name	(#constrs, #vars)	[nnz(A), nnz(M)]
(a) Statistics of the NEPs		
vimod1	(554, 1138)	(4744, 22,577)
vimod2	(910, 1723)	(7935, 46,137)
vimod3	(1101, 2226)	(9117, 67,634)
vimod4	(870, 1828)	(62,056, 154,332)
vimod5	(1327, 2586)	(133,527, 274,004)
vimod6	(2210, 4359)	(207,408, 417,810)

Name	Number of iterations			Elapsed time (s)		
	PATHAVI	PATH	PATHAVI/ UMFPACK	PATHAVI	PATH	PATHAVI/ UMFPACK
(b) # Iterations and elapsed time of PATHAVI and PATH on the NEPs						
vimod1	367	2087	367	0.372	4.129	0.437
vimod2	319	3570	319	1.098	24.134	0.645
vimod3	590	4278	590	3.208	60.553	1.639
vimod4	1343	6146	1343	127.194	66.427	18.319
vimod5	2167	2768	2167	327.970	325.558	40.285
vimod6	3522	4222	3522	2341.193	1841.642	109.960

In this case, x is a solution to (NEP) if and only if it is a solution to the AVI(C, q, M) where $Mx + q = (\nabla_{x_i} h_i(x))_{i=1}^N$ and $C = \Pi_{i=1}^N X_i$. The number of agents ranges from 10 to 300.

Table 4 presents performance of PATHAVI and PATH over the NEPs. The number of iterations of PATHAVI is up to 11 times fewer than PATH. Elapsed time shows similar results except for the last three instances. In those instances, LUSOL has a great difficulty in computing PATHAVI's intermediate basis matrices. If we change the linear algebra engine to UMFPACK, the computation time significantly reduces. Regarding PATH's performance on the last three instances, we would like to point out that the proximal perturbation technique of PATH, which solves a sequence of perturbed MCPs by adding $\epsilon_k I$ with $\epsilon_k \to 0$ as $k \to \infty$ to the matrix $\tilde{M}$ in (MCP-reform), plays a significant role in its performance. Adding positive diagonals elements changes the elimination sequence and makes linear algebra computations much faster and more stable. When we turn off the proximal perturbation, PATH either gets much slower than PATHAVI or fails to solve the instance.

7 Conclusions

We have presented PATHAVI, a structure-preserving pivotal method for affine variational inequalities. Compared to existing methods, PATHAVI can process an AVI without applying any reduction or transformation to the problem data even if the underlying feasible region contains lines. PATHAVI can process some newly generated problem classes from applications in friction contact as well as the existing problem class (L-matrices [6]). A computational method for finding a point satisfying sufficient conditions for a ray start is detailed. Through worst-case analysis, we have shown that exploiting polyhedral structure for solving affine variational inequalities is expected to show better performance than using a mixed complementarity problem reformulation. Computational results over friction contact and Nash equilibrium problems illustrate that PATHAVI compares favorably with PATH both in terms of robustness and efficiency.

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Appendix

Lemma 5 (Theorem 4.4 [6]) *Consider an AVI*(C, q, M) *with* $\text{lin } C = \{0\}$ *and let* M *be semimonotone with respect to* $\text{rec } C$. *Suppose that an unbounded ray occurs. Then the value of the auxiliary variable* t *is constant on that ray and* Δz, *the variation in* z, *is nonzero and satisfies*

$$\Delta z \in \text{rec } C, \qquad M\Delta z \in (\text{rec } C)^D, \qquad \textit{and} \qquad \Delta z^T M \Delta z = 0. \tag{14}$$

Proof The fact that t is constant and that Δz is a solution to (14) follows from the first part of the proof of Theorem 4.4 in [6]. To see that the direction Δz is nonzero, we proceed by contradiction: at the current iterate (x^k, t^k) we have

$$G_C(x^k, t^k) = Mz^k + q + x^k - z^k - t^k r = 0. \tag{15}$$

Let x^{k+1} belong to the unbounded ray and suppose that $\Delta z = 0$:

$$G_C(x^{k+1}, t^k) = Mz^k + q + x^{k+1} - z^k - t^k r = 0.$$

It immediately follows that $x^{k+1} = x^k$. □

Algorithm 1 Pivoting to make as many nonbasic free variables as basic variables

Input: a basic feasible solution z^0 and its index sets $(B^0, N_l^0, N_u^0, N_{fr}^0, \mathscr{A}^0, \bar{\mathscr{A}}^0)$
Output: a basic feasible solution $\bar{z}^0$ and its index sets $(B, N_l, N_u, N_{fr}, \mathscr{A}, \bar{\mathscr{A}})$
Set $\bar{z}^0 \leftarrow z^0$.
Set $(B, N_l, N_u, N_{fr}, \mathscr{A}, \bar{\mathscr{A}}) \leftarrow (B^0, N_l^0, N_u^0, N_{fr}^0, \mathscr{A}^0, \bar{\mathscr{A}}^0)$.
Set `changed` $\leftarrow$ `true`.
while `changed` is `true` **do**
    Set `changed` $\leftarrow$ `false`.
    for each $j \in N_{fr}$ **do**
        Do a ratio test on the nonbasic column j over basic variables that are not free variables.
        if the ratio is finite **then**
            Pivot in the jth column into basis.
            Update $\bar{z}^0$ and its index sets $(B, N_l, N_u, N_{fr}, \mathscr{A}, \bar{\mathscr{A}})$. $\triangleright\ |N_{fr}| \leftarrow |N_{fr}| - 1$
            Set `changed` $\leftarrow$ `true`.
        end if
    end for
end while
return $\bar{z}^0$ and its index sets $(B, N_l, N_u, N_{fr}, \mathscr{A}, \bar{\mathscr{A}})$

Algorithm 2 Overall computation procedure of PATHAVI

Input: AVI(C, q, M)
Output: One of the following: emptiness of C, a solution z^* to the AVI(C, q, M), or a secondary ray
Construct and solve the LP problem defined in (LP) using an LP solver.
if the LP solver determines that C is empty **then**
    return C is empty
end if
Let z^0 be the basic feasible solution returned by the LP solver.
Run Algorithm 1 with z^0 and its index sets to compute an implicit extreme point $\bar{z}^0$.
Construct and solve the complementary system of equations defined in Proposition 10 using $\bar{z}^0$ and its index sets.
if $(\bar{z}^0, \lambda, w, v, s)$ is feasible **then**
    Set $z^* \leftarrow \bar{z}^0$.
    return z^*.
else
    Choose $r \in \mathrm{ri}\,(N_C(\bar{z}^0))$ by referring to the active constraint set at $\bar{z}^0$.
    Augment a column $(r, 0)^T$ with a t variable to the complementary system of equations.
    Compute an almost complementary feasible basis by pivoting in the t variable.
    Do complementary pivoting until either we find a solution z^* or a secondary ray is generated.
    return z^* if we have found a solution or a secondary ray otherwise.
end if

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