Fall 2014 Qualifier Exam:
OPTIMIZATION

September 15, 2014

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 4 of 5 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Let $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, $i = 1, \ldots, m$, $d > 0$, and $c \in \mathbb{R}^n$.

   (a) Formulate the problem

$$\max\Big\{c^\top x : \sum_{i=1}^{m} \max\{a_i^\top x - b_i, 0\} \le d, x \ge 0\Big\} \tag{1}$$

   as a compact linear program. (Compact here means that the number of decision variables and constraints is polynomial in n and m.)

   (b) Now write down an (exponential) number of linear inequalities just involving x that would be equivalent to the original nonlinear inequality in (1), and show this equivalence. (Hints: (i) To build intuition on what the form of this formulation will be, you may find it helpful to consider a special case with $n = 1$, and also start with $m = 1$ and $m = 2$. (ii) Alternatively, it may help to think about how you would check if a given solution $\hat{x}$ is feasible to (1).)

   (c) Explain how you would use a cutting plane approach for solving the formulation defined using the inequalities in part (b).

Answer:

(a) Introduce variables $y_i, i = 1, \ldots, m$ yielding the formulation:

$$\begin{aligned} &\max\ c^\top x \\ &\text{s.t.} y_i \ge a_i^\top x - b_i, \imath = 1, \ldots, m \\ &\sum_{i=1}^{m} y_i \le d \\ &y \ge 0, x \ge 0 \end{aligned}$$

(b) The formulation is:

$$\begin{aligned} &\max\ c^\top x \\ &\text{s.t.} \sum_{i\in S} (a_i^\top x - b_i) \le d, \quad \forall S \subseteq \{1, \ldots, m\} \\ &x \ge 0 \end{aligned} \tag{2}$$

Suppose x is feasible to (1) and let $S \subseteq \{1, \ldots, m\}$. Then

$$\sum_{i\in S} (a_i^\top x - b_i) \le \sum_{i\in S} \max\{a_i^\top x - b_i, 0\} \le \sum_{i=1}^{m} \max\{a_i^\top x - b_i, 0\} \le d$$

and therefore x is feasible to (2).

Now suppose x is feasible to (2). Then, let $S^* = \{i = 1, \ldots, m : a_i^\top x - b_i > 0\}$. By definition of S^* and (2), it holds that

$$\sum_{i=1}^{m} \max\{a_i^\top x - b_i, 0\} = \sum_{i\in S^*} (a_i^\top x - b_i) \le d.$$

(c) Separation of a violated inequality at a point $\hat{x}$ is efficient: just find $\hat{S} = \{i : a_i^\top \hat{x} - b_i > 0$ and evaluate $\sum_{i \in \hat{S}} (a_i^\top \hat{x} - b_i)$. If it exceeds d a violated inquality is found, otherwise there is none. This separation can either be used in the ellipsoid algorithm for a polynomial algorithm, or in conjunction with the simplex algorithm as part of a cutting plane algorithm that is likely to solve efficiently in practice.

◇

2. Consider the following data describing hydrological characteristics of a small hydroelectric power station.

m	denotes month
f_m	Water inflow in month m (million cubic meters)
$\bar{p}_m$	Market price of electricity in month m
L_{max}	Maximum water level the dam can store (million cubic meters)
L_{min}	Minimum water level the dam can store (million cubic meters)
R_{max}	Maximum water which can be released per month
κ	Energy per amount of water (megawatt hours per million cubic meters)

In any given month, water may be spilled to respect the maximum reservior level. When water is spilled, it leaves the reservoir without producing energy.

(a) Formulate a *steady-state* monthly linear programming (LP) model which maximizes annual profit, taking market prices as given.

Use the following notation:

L_m Reservior level at the start of month m

R_m Water released during month m to generate electricity

S_m Water spilled during month m

(b) Write out key elements of the GAMS code for this model.

(c) Suppose that monthly demand as a function of price (p_m) is given by:

$$D_m = \alpha_m - \beta_m p_m$$

where p_m is the market price, and α_m and β_m are both positive constants. Formulate a *quadratic programming* (QP) model to determine the production profile which maximizes profit.

Answer:

(a) Let R_m denote the water released to generate electricity in month m, and let S_m represent the water which is spilled, and let L_m represent the reservior level at the start of month m.

$$\max \sum_m \bar{p}_m \kappa R_m$$

subject to:

$$L_{m++1} = L_m + f_m - R_m - S_m$$

$$L_{min} \leq L_m \leq L_{max}$$

$$R_m \leq R_{max}$$

$$R_m, S_m \geq 0$$

Here we have used the "++" operator to reference the set of months on a "ring", i.e. $january = december + +1$.

(b) The GAMS code is:

(c)

$$\max \sum_m D_m(\alpha_m - D_m)/\beta_m$$

subject to:

$$D_m = \kappa R_m$$

$$L_{m++1} = L_m + f_m - R_m - S_m$$

$$L_{min} \leq L_m \leq L_{max}$$

$$R_m \leq R_{max}$$

$$R_m, S_m \geq 0$$

$\diamond$

3. Let $n \in \mathbb{Z}$ with $n \geq 2$, and for all $1 \leq i < j \leq n$ consider the following sets of constraints:

$$x_i + x_j - y_{ij} \leq 1, \tag{3}$$

$$-x_i + y_{ij} \leq 0, \tag{4}$$

$$-x_j + y_{ij} \leq 0, \tag{5}$$

$$-y_{ij} \leq 0, \tag{6}$$

$$x_i \text{ integer}, \quad y_{ij} \text{ integer}, \tag{7}$$

We denote by QP^n_{LP} the polyhedron defined by $(3), \ldots, (6)$:

$$QP^n_{LP} = \{(x, y) \in \mathbb{R}^{n(n+1)/2} : (x, y) \text{ satisfies } (3), \ldots, (6)\},$$

and by QP^n the integer hull of QP^n_{LP}:

$$QP^n = \text{conv}\{(x, y) \in \mathbb{R}^{n(n+1)/2} : (x, y) \text{ satisfies } (3), \ldots, (7)\}.$$

(a) Show that if $(x, y) \in QP^n_{LP}$, then $0 \leq x_i \leq 1$ and $y_{ij} \leq 1$ for all $1 \leq i < j \leq n$.

(b) What is the dimension of QP^n?

(c) Prove or disprove that $QP^2 = QP^2_{LP}$.

(d) Show that $QP^3 \neq QP^3_{LP}$ by giving a fractional vertex of QP^3_{LP}. Give a Gomory-Chvátal Rounding inequality that cuts off such fractional vertex.

Answer:

(a) Inequalities $x_i \geq 0$ can be obtained by summing (4) and (6), inequalities $x_i \leq 1$ can be obtained by summing (3) and (5), inequalities $y_{ij} \leq 1$ can be obtained by summing (3), (4), and (5).

(b) We show that QP^n is full-dimensional by giving $n(n + 1)/2 + 1$ affinely independent vectors that satisfy constraints $(3), \ldots, (7)$:

  (i) The zero vector (1 vector);

  (ii) For every $1 \leq i \leq n$, the vector (x, y) with $x_i = 1$, and all other components equal to zero (n vectors);

  (iii) For every $1 \leq i < j \leq n$, the vector (x, y) with $x_i = x_j = y_{ij} = 1$, and all other components equal to zero ($\binom{n}{2}$ vectors).

(c) The polytope QP^2_{LP} has dimension 3 and is defined by 4 inequalities. Therefore it can be easily checked that its vertices are all integral: $(0,0,0)$, $(0,1,0)$, $(1,0,0)$, $(1,1,1)$. Therefore $QP^2 = QP^2_{LP}$.

(d) A fractional vertex $(\bar{x}, \bar{y})$ of QP^3_{LP} is given by $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 1/2$, and $\bar{y}_{ij} = 0$, $1 \leq i < j \leq 3$. This can be seen as it is the unique vector that satisfies tightly the 6 inequalities (3) and (6). The vector $(\bar{x}, \bar{y})$ is cut off by the inequality $x_1 + x_2 + x_3 - y_{12} - y_{13} - y_{23} \leq 1$, which is the Gomory-Chvátal rounding inequality obtained by summing inequalities (3) and (6), dividing the resulting inequality by 2, and then rounding down the right-hand side.

$\diamond$

4. Let $C = \{(x_1, x_2) : -x_1 + 2x_2 \le 0, -x_1 - 2x_2 \le 0\}$, i.e. $2\,|x_2| \le x_1$.

   (a) Define the normal cone to C at x (in the general case for a convex set C) and determine $N_C(x)$ for every $x \in \mathbf{R}^2$ in this specific example.

   (b) Consider the problem
   $$\min_{x\in C} \frac{1}{2}(x_1^2 - x_2^2) - px_1$$
   Show that for $p = 0$ the origin is a strict local minimizer of this problem. (If you use the second order sufficient conditions, be careful to define these precisely and define the sets that are used in its statement).

   (c) Now let p assume small positive values. How many stationary points (points satisfying the first order necessary conditions) are there near the origin? What are they? What kinds of points are they (local minimizers, saddle points, local maximizers)?

   (d) Suppose we change C to $\{(x_1, x_2) : 2x_2^2 \le x_1\}$. How does the answer to (b) change?

Answer:

(a) Faces are C, $\{(0, 0)\}$, $F_1 = \{(x_1, x_2) : -x_1 + 2x_2 = 0, x_1 \ge 0\}$, $F_2 = \{(x_1, x_2) : -x_1 - 2x_2 = 0, x_1 \ge 0\}$. If $x \notin C$, then $N_C(x) = \emptyset$. For $x \in \operatorname{rint} C$, $N_C(x) = \{(0, 0)\}$. For $x \in \operatorname{rint} F_1$, $N_C(x) = \{(-\lambda_1, 2\lambda_1) : \lambda_1 \ge 0\}$. For $x \in \operatorname{rint} F_2$, $N_C(x) = \{(-\lambda_2, -2\lambda_2) : \lambda_2 \ge 0\}$. For $x = (0, 0)$, $N_C(x) = \{(-\lambda_1, 2\lambda_1) + (-\lambda_2, -2\lambda_2) : \lambda \ge 0\}$. (Note that $N_C(x)$ is constant on the relative interiors of the faces of C.)

(b) The critical cone is $K_C(x) = T_C(x) \bigcap [\nabla f(x)]^{\perp}$. SOSC is FONC and $\nabla^2 f(x)$ is positive definite on $K_C(x)$.
$$\nabla^2 f(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
and $K_C(0) = C$ and thus $0 \ne d \in C$ has $d^T \nabla^2 f(0) d = d_1^2 - d_2^2$ which is strictly positive if $d_2 = 0$, and $\ge 3d_2^2 > 0$ otherwise.

(c) When $p > 0$, the origin is no longer stationary ($0 \notin \nabla f(0) + N_C(0)$). There are three stationary points $(4p/3, \pm 2p/2)$, $(p, 0)$. The first two are strict local minimizers and the third is a saddle point. (Note that $T_C(x)$ easily calculated from $N_C(x)$ given above and hence the critical cone is easy to write down).

(d) At $p = 0$, origin becomes saddle point since moving along the curved boundary decreases f. (Details: $T_C(0) = \mathbf{R}_+ \times \mathbf{R}$, and hence Hessian is only positive semidefinite on $K_c(0)$, so SOSC not satisfied. If we take $x(\mu) = (2\mu^2, \mu) \in C$ then $f(x(\mu)) = 0.5(4\mu^4 - \mu^2) < 0$ for μ small.) (While not needed here, note that for $p > 0$, there are still 3 stationary points $(p + 1/4, \pm\sqrt{p/2 + 1/8})$, $(p, 0)$, the first two are strict local minimizers and the third is a saddlepoint.)

$\diamond$

5. (a) Consider the following constrained optimization problem:

$$\min_{x \in \mathbf{R}^n} f(x) \text{ subject to } c_i(x) = 0,\ i = 1, 2, \ldots, m,\ \ h_j(x) \geq 0,\ j = 1, 2, \ldots, r,$$

where the functions f, c_i, $i = 1, 2, \ldots, m$, and h_j, $j = 1, 2, \ldots, r$ are all continuously differentiable. Write down KKT necessary conditions for optimality of a point x^*.

(b) Write down the linear independence constraint qualification (LICQ) and Mangasarian-Fromovitz constraint qualification (MFCQ) for the problem in (a) at the point x^*.

(c) Consider the Lagrange multipliers which, along with the point x^*, satisfies the KKT conditions for the problem in part (a). Denote these multipliers by λ_i^*, $i = 1, 2, \ldots, m$ for the equality constraints and μ_j^*, $j = 1, 2, \ldots, r$ for the inequality constraints. Show that when LICQ holds, the set of multipliers satisfying the KKT conditions contains a single point.

(d) Consider the problem with inequality constraints only (that is, $m = 0$), and let μ_j^*, $j = 1, 2, \ldots, r$ be optimal Lagrange multipliers for the inequality constraints, as in part (c). Show that when MFCQ holds, this set of multipliers is bounded. (Hint: Assume for contradiction that $\{\mu^k\}_{k=1,2,\ldots} = \{(\mu_1^k, \mu_2^k, \ldots, \mu_r^k)^T\}_{k=1,2,\ldots}$ is a sequence such that each μ^k is a vector of optimal Lagrange multipliers for the inequality-constrained problem such that $\lim_k \|\mu^k\| = \infty$. Consider limit points $\bar{\mu}$ of the sequence of unit vectors $\{\mu^k / \|\mu^k\|\}$.)

(e) Consider the following nonlinear program with complementarity constraint:

$$\min_{x \in \mathbb{R}^n} f(x) \text{ subject to } g_1(x) \geq 0,\ g_2(x) \geq 0,\ g_1(x) g_2(x) = 0,$$

where f, g_1, and g_2 are all continuously differentiable functions that map $\mathbb{R}^n$ to $\mathbb{R}$. Show that the LICQ and MFCQ cannot be satisfied at any feasible point of this problem.

Answer:

(a) KKT conditions are that there exist multipliers λ_i^*, $i = 1, 2, \ldots, m$ and μ_j^*, $j = 1, 2, \ldots, r$ such that

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla c_i(x^*) - \sum_{j=1}^{r} \mu_j^* \nabla h_j(x^*) = 0, \tag{8a}$$

$$c_i(x^*) = 0, \qquad i = 1, 2, \ldots, m, \tag{8b}$$

$$0 \leq \mu_j^* \perp h_j(x^*) \geq 0, \quad j = 1, 2, \ldots, r. \tag{8c}$$

(b) Denote by $\mathcal{A}^*$ the set of active inequality constraints at x^*, that is,

$$\mathcal{A}^* := \{j = 1, 2, \ldots, r \mid h_j(x^*) = 0\}.$$

LICQ is that the set of vectors $\{\nabla c_i(x^*) \,|\, i = 1, 2, \ldots, m\} \cup \{\nabla h_j(x^*) \,|\, j \in \mathcal{A}^*\}$ is linearly independent. MFCQ is that (i) the set of vectors $\{\nabla c_i(x^*) \,|\, i = 1, 2, \ldots, m\}$ is linearly indepedent and (ii) there exists a vector $v \neq 0$ such that

$$\nabla c_i(x^*)^T v = 0, \; i = 1, 2, \ldots, m; \;\; \nabla h_j(x^*)^T v > 0, \; j \in \mathcal{A}^*.$$

(c) If LICQ holds, uniqueness of the optimal Lagrange multipliers follows immediately from the following modification of (8a):

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla c_i(x^*) - \sum_{j \in \mathcal{A}^*} \mu_j^* \nabla h_j(x^*) = 0. \tag{9}$$

(d) To show boundedness of optimal multipliers for MFCQ, we argue by contradiction, using the unbounded sequence of multiplers $\{\mu^k\}$ defined in the hint. Without loss of generality (by taking a subsequence if necessary) and using compactness of the unit ball, we can identify a vector $\bar{\mu}$ with $\|\bar{\mu}\| = 1$ such that

$$\mu^k / \|\mu^k\| \to \bar{\mu}.$$

Since $\mu_j^k = 0$ for $j \notin \mathcal{A}^*$, we have $\bar{\mu}_j = 0$ for $j \notin \mathcal{A}^*$. Additionally, it is clear that $\bar{\mu}_j \geq 0$ for all $j \in \mathcal{A}^*$. From (9), we have

$$\nabla f(x^*) - \sum_{j \in \mathcal{A}^*} \mu_j^k \nabla h_j(x^*) = 0, \quad \text{for all } k. \tag{10}$$

By dividing (10) by $\|\mu^k\|$ and taking limits as $k \to \infty$, we have

$$\sum_{j \in \mathcal{A}^*} \bar{\mu}_j \nabla h_j(x^*) = 0. \tag{11}$$

Taking the inner product of both sides with the vector v in the definition of MFCQ, we have

$$0 = \sum_{j \in \mathcal{A}^*} \bar{\mu}_j (v^T \nabla h_j(x^*)).$$

Since $v^T \nabla h_j(x^*) > 0$ for $j \in \mathcal{A}^*$ and $\bar{\mu}_j \geq 0$ for all $j \in \mathcal{A}^*$, this last expression can be satisfied only if $\bar{\mu}_j = 0$ for $j \in \mathcal{A}^*$. This contradicts $\|\bar{\mu}\| = 1$. We conclude that the set of multipliers satisfying the KKT conditions must be bounded.

(e) The gradient of the equality constraint is

$$g_1(x) \nabla g_2(x) + g_2(x) \nabla g_1(x).$$

For any feasible point we have three possible scenarios:

(i) $g_1(x) = 0$ and $g_2(x) > 0$. In this case the active inequality constraint gradient is $\nabla g_1(x)$ while the equality constraint gradient is $g_2(x) \nabla g_1(x)$. These two vectors are linearly dependent, so LICQ is not satisfied. For MFCQ to be satisfied, we would need a vector v such that both $\nabla g_1(x)^T v > 0$ and $g_2(x) \nabla g_1(x)^T v = 0$. Since $g_2(x) = 0$, these conditions are incompatible, so MFCQ cannot be satisfied either.

(ii) $g_1(x) > 0$ and $g_2(x) = 0$. The argument here is identical to (i), with the indices "1" and "2" reversed.

(iii) $g_1(x) = g_2(x) = 0$. In this case the gradient of the inequality constraint is zero, so neither LICQ nor MFCQ can be satisfied.

$\diamond$