Compressing Branch-and-Bound Trees

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Branch and Bound Trees

Branch and bound (BB) tree:

- Each node v corresponds to $Q(v)$
- Each non-leaf node v has children:

$$Q(v) \cap \{x : \pi^T x \leq \pi_0\} \text{ and}$$

$$Q(v) \cap \{x : \pi^T x \geq \pi_0 + 1\}$$

where $\pi \in \mathbb{Z}^n, \pi_0 \in \mathbb{Z}$.

Tree dual bound:

$$d(T, c) = \min_{v \in L(T)} \min \left\{ c^T x : x \in Q(v) \right\}$$

Previous Research

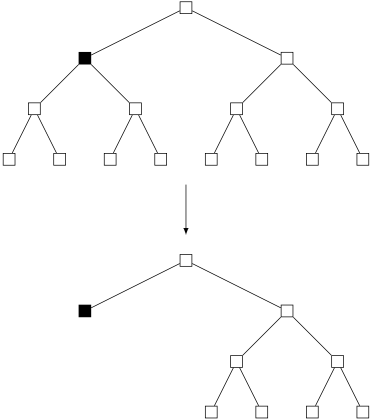
- **Variable branching rules:**
  - Pseudocost branching — Benichou et al (1971)
  - Strong branching — Applegate, Bixby, Chvátal & Cook (1995)
  - Reliability branching — Achterberg, Koch & Martin (2005)
- **Branching on general directions:**
  - Owen & Mehrotra (2001)
  - Mahajan & Ralphs (2009)
  - Cornuejols, Liberti, Nannicini (2011)
  - Gamrath & al (2015)
- **Bounds on tree size:**
  - Exponential size with var. disjunctions — Jeroslow (1974), Chvatal (1980)
  - Exponential size with general disjunctions — Dadush et al. (2020), Dey et al. (2022)
  - Size under limited support size — Basu, Conforti, Di Summa & Jiang (2021)
  - Full strong branching tree size — Dey, Dubey & Molinaro (2022)

Work Overview

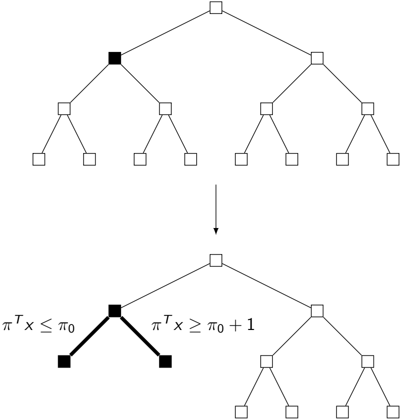
- **Research question:** Can we make a BB tree smaller without deteriorating dual bound?
- **Motivation:**
  - Small dual certificates
  - Strong disjunctions for instance families
  - Training data for ML branching methods
- **Talk outline:**
  1. Tree Compression Problem (TCP)
  2. Complexity & lower bound results
  3. Exact & heuristic algorithms
  4. MIPLIB 3 & 2017 computational experiments

Tree Operations

drop(T, v)



replace(T, v, π, π_0)



The Tree Compression Problem

Compression: T_k is a compression of T_1 if $\exists T_1, T_2, \ldots, T_k$ such that:

1. $T_i = \text{drop}(T_{i-1}, v)$ or $T_i = \text{replace}(T_{i-1}, v, \pi, \pi_0)$; and
2. $|T_i| < |T_{i-1}|$; and
3. $d(T_i, c) \geq d(T_{i-1}, c)$

Tree Compression Problem (TCP): Given a branch and bound tree T, an objective vector c, and a set of branching directions $\mathcal{D}$, is there a compression T' of T?

NP-Completeness I

Disjunctive Infeasibility (DI):

- Let $S = \{x \in \mathbb{R}^n : Ax \leq b\}$ where $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$.
- Does there exist $\pi \in \mathbb{Z}^n \setminus \{0\}, \pi_0 \in \mathbb{Z}$ such that:

$$S \subseteq \{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}?$$

Mahajan & Ralphs (2009): (DI) is NP-complete.

Theorem 1. (TCP) is NP-complete when $\mathcal{D} = \mathbb{Z}^n$ and $c = 0$.

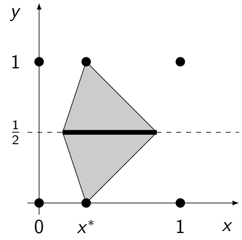
NP-Completeness II

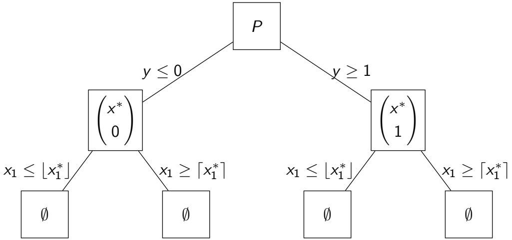
Proof Sketch:

1. Let S be an instance of (DI)
2. Let $x^* \in S \setminus \mathbb{Z}^n$. WLOG $x_1 \notin \mathbb{Z}$.
3. Let

$$P = \operatorname{conv}\left(\left\{\begin{pmatrix} x^* \\ 0 \end{pmatrix}, \begin{pmatrix} x^* \\ 1 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} x \\ \frac{1}{2} \end{pmatrix} : x \in S\right\}\right)$$

4. Build the tree on the right.
5. If (DI) has a YES answer (π, π_0) then replace$(T, r, (\pi, 0), \pi_0)$ is a compression.
6. If tree is compressible, it must be with replace$(T, r, (\pi, \pi_{n+1}), \pi_0)$, where $\pi_{n+1} = 0$ and r is the root.





Additional Results

Theorem 2. There exists a tree T with root polyhedron $P \subseteq R^{n+1}$ such that:

1. $|T| \geq 2^{n+1}$ and $d(T, 0) = \infty$
2. Best compression of T has at least $\frac{2^n-1}{n}$ nodes
3. There exists T' with root P s.t. $|T'| = 7$ and $d(T', 0) = \infty$.

Proposition: Suppose T is generated with full strong branching and best bound on directions $D \subseteq \mathbb{Z}^n$. Let T' be a compression of T using the same directions. Then:

1. Dual bound does not improve: $d(T, c) = d(T', c)$
2. Drop operation is sufficient

Exact Algorithm

Observation: replace(T, v, π, π_0) is a compression of T if and only if:

$$\min\{c^T x : x \in Q(v), \pi^T x \leq \pi_0\} \geq d(T, c) \text{ and}$$

$$\min\{c^T x : x \in Q(v), \pi^T x \geq \pi_0 + 1\} \geq d(T, c)$$

MILP Formulation [Mahajan & Ralphs (2009)]:

$$\max_{\substack{\delta, \mathrm{p}, \mathrm{q}, \boldsymbol{\pi}, \\ \pi_0, s_L, s_R}} \left\{ \delta : \begin{array}{ll} \mathrm{A}^\top \mathrm{p} - s_L \mathrm{c} - \boldsymbol{\pi} = 0, & \mathrm{p}^\top \mathrm{b} - d(T, \mathrm{c}) s_L - \pi_0 \geq \delta \\ \mathrm{A}^\top \mathrm{q} - s_R \mathrm{c} + \boldsymbol{\pi} = 0, & \mathrm{q}^\top \mathrm{b} - d(T, \mathrm{c}) s_R - \pi_0 \geq \delta - 1 \\ \mathrm{p}, \mathrm{q} \geq 0, \ s_L, s_R \geq 0, & \boldsymbol{\pi} \in \mathbb{Z}^n, \ \pi_0 \in \mathbb{Z} \end{array} \right\}$$

Exact Algorithm: Solve MILP for every node.

Heuristic Algorithm

Heuristic for General Branching Directions:

- Owen & Mehrotra (2001)
- Cornuejols, Liberti, Nannicini (2011)
- Karamanov & Cornuejols (2011)
- Mahmoud & Chinneck (2013)
- Gamrath et al. (2015)

Owen & Mehrotra's Heuristic:

- Find best single variable direction (π, π_0)
- For each fractional x_i^* consider $\pi + e_i$ and $\pi - e_i$.
- Repeat until no further improvement

MIPLIB 3 Experiments: Setup

Questions

1. How compressible are realistic BB trees?
2. How much compression is achievable in short running times?

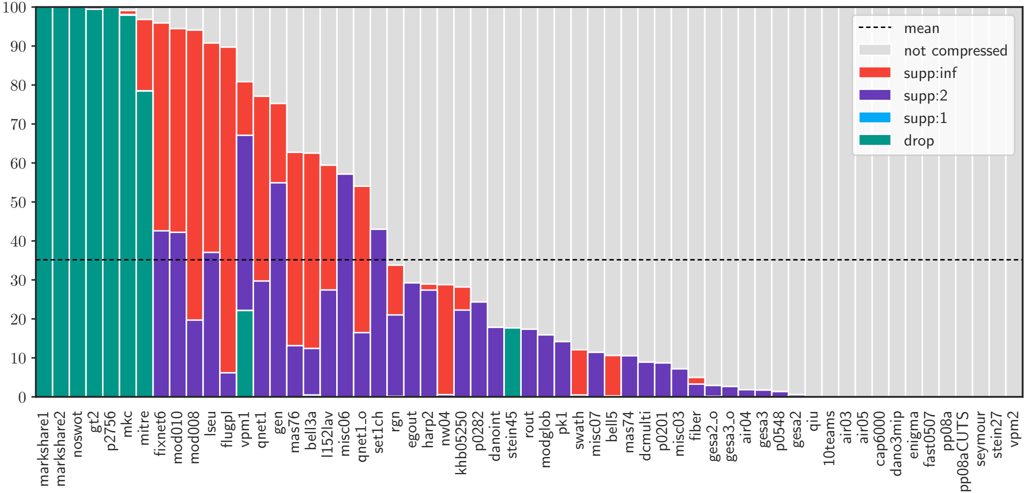
Branching rules considered:

1. Full strong branching (FSB)
2. Reliability branching (RB) with plunging

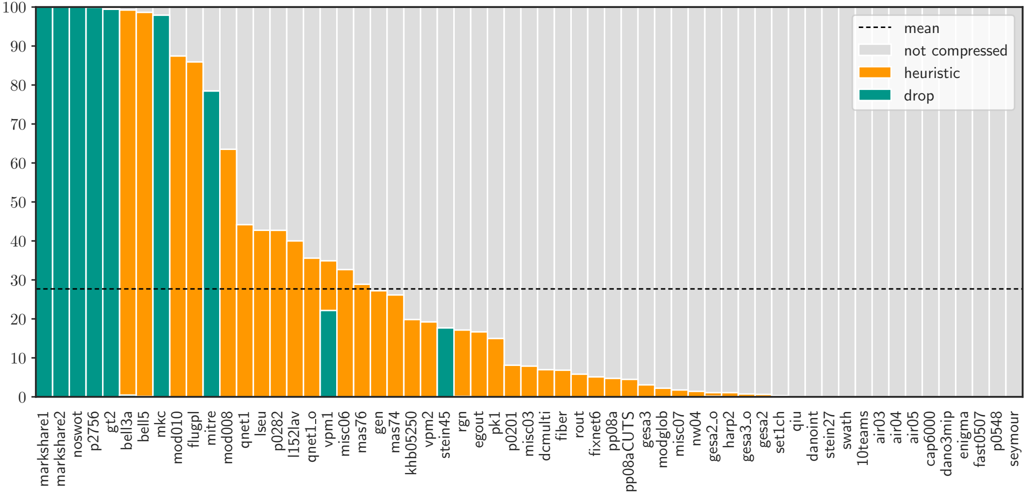
Implementation & Environment:

1. Julia, JuMP, Gurobi 9.5
2. MIPLearn: Custom B&B Implementation
3. Tree generation: 10k node limit, no time limit
4. Tree compression: 24-hour limit for exact, 15-minute for heuristic
5. AMD Ryzen 9 7950x (5.7GHz, 16C, 32T, 128 GB RAM)

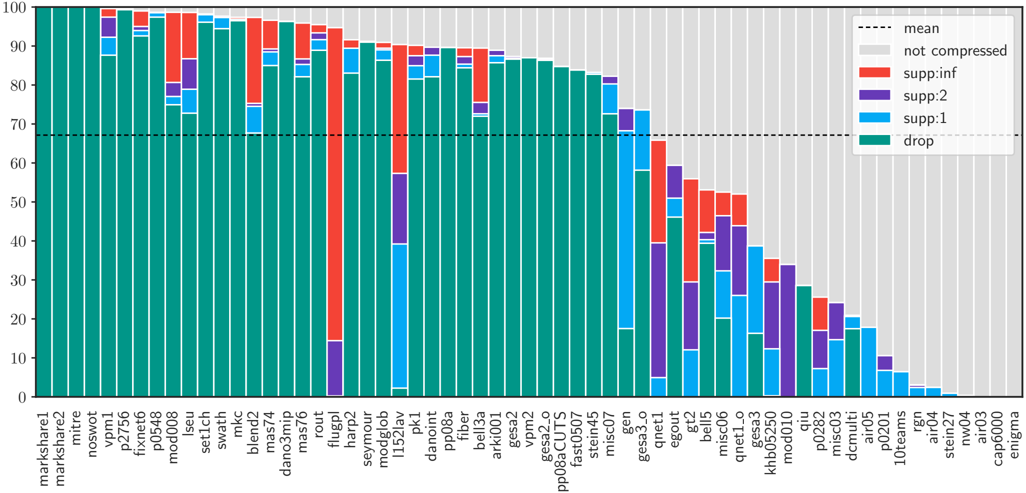
MIPLIB 3 Experiments: FSB/Exact



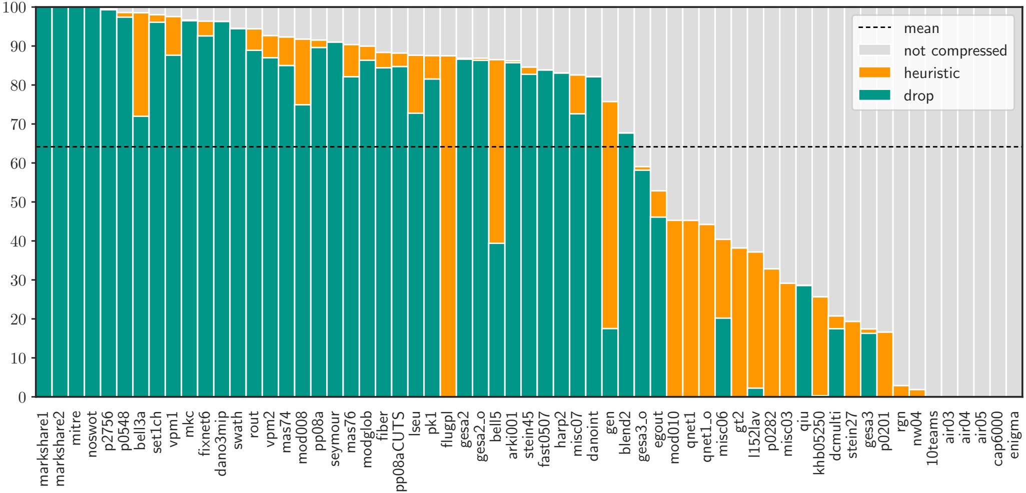
MIPLIB 3 Experiments: FSB/Heuristic



MIPLIB 3 Experiments: RB/Exact



MIPLIB 3 Experiments: RB/Heuristic



MIPLIB 2017: Setup

Challenge: Node subproblems become too expensive

Node orderings:

1. `Random`
2. `DFS`
3. `NodeId`
4. `SubtreeSize`
5. `Gap`
6. `Expert`

Implementation:

- Precomputed compressibility info
- Reliability branching without plunging

MIPLIB 2017: Results

Node Ordering	AUC (%)	Compression Ratio (%)		
	1-hour	15-min	1-hour	4-hour
Expert	65.4	30.8	34.4	35.1
Gap	76.4	18.2	25.7	30.5
NodeId	79.6	15.5	21.6	27.9
SubtreeSize	79.6	15.6	21.7	28.1
Random	80.7	13.3	21.2	28.7
DFS	83.3	12.9	17.1	24.0

Conclusion & Future Work

In this talk:

- Tree compression problem
- NP-completeness and bound results
- Algorithms and MIPLIB experiments

Future work:

- Provably compressible trees
- Better heuristics
- Use directions found to accelerate MIP

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