# A Deterministic Better-than-3/2 Approximation Algorithm for the Metric Traveling Salesperson Problem


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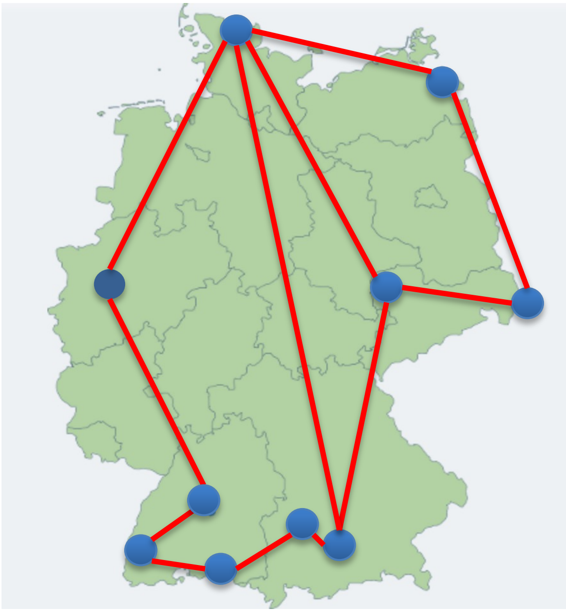
# Metric, symmetric TSP

Given a set of cities and their pairwise symmetric distances satisfying the triangle inequality,

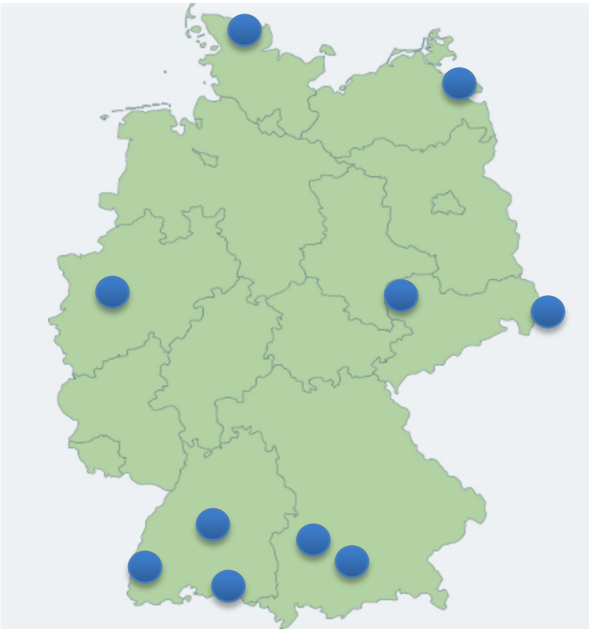
$$d(u, v) \leq d(u, w) + d(w, v)$$

find the minimum cost tour that visits every city at least once.

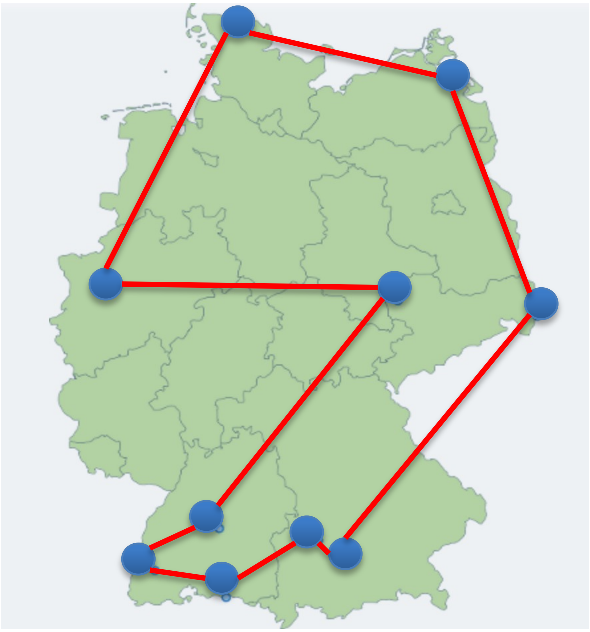
In other words, given a weighted graph, find a **minimum cost spanning Eulerian subgraph**.



# Approximation



Approximate tour
(efficiently computed)



*Best approximation algorithm:* at most **50% worse** than optimal, i.e. a 1.5-approximation [Christofides ’76, Serdyukov ‘78]

*Lower bounds:* can’t do better than **about 1%** (currently 123/122) unless P=NP [Papadimitriou-Yannakakis ‘93, Böckenhauer-Seibert ‘00, Papadimitriou-Vempala ’00, Engebretsen ‘03, Lampis ‘12, Karpinski-Lampis-Schmied ’15]

# Approximation algorithms

[Christofides '76, Serdyukov ‘78]: 3/2 approximation

[Wolsey ‘80, Shmoys-Williamson ‘90]: 3/2 integrality gap of LP relaxation

[Arora ‘96, Mitchell ‘96]: PTAS for Euclidean TSP

[Papadimitriou-Yannakakis ‘93,Blaser-Ram ‘05,Berman-Karpinski ‘06]: 1.14 for (1,2) TSP

[Grigni-Koutsoupias-Papadimitriou ’95, Arora-Grigni-Karger-Klein-Woloszyn ‘98, Klein ‘05]: PTAS for planar TSP

[Talwar ‘04, Bartal-Gottlieb-Krauthgamer ‘12]: PTAS for TSP on metrics with bounded doubling dimension.

[Gamarnik-Lewenstein-Sviridenko ‘05, Aggarwal-Garg-Gupta ‘11, Boyd-Sitters-Ster- Stougie ‘11, Correa, Larre, Soto '12]: 4/3 and even below for graphic TSP on (sub)cubic graphs.

[Demaine-Hajiaghayi-Mohar’07, Demaine-Hajiaghayi-Kawarabayashi ‘11]: PTAS for TSP on bounded genus and minor free graphs

[Oveis Gharan-Saberi-Singh ‘10] [Mömke-Svensson ‘11] [Mucha ‘11] [Sebő-Vygen ’12]: 1.4 for graphic TSP

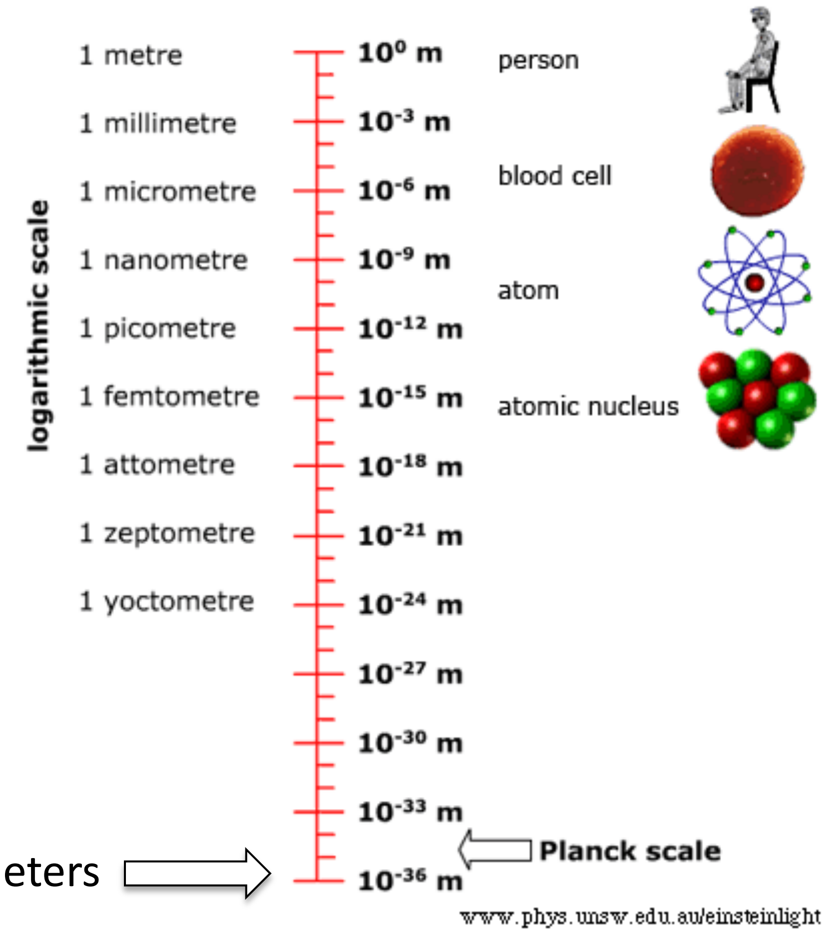
[Carr-Ravi ’98, Boyd-Carr ’11,Boyd-Legault ’15, Boyd-Sebő ’17, Haddadan-Newman-Ravi ’18, Hadaddan- Newman ’19, Karlin-K-Oveis Gharan ‘20][Gupta-Lee-Li-Mucha-Newman-Sarkar ‘21]: 1.4983 for half integral TSP

[Hoogeveen ‘91][An-Kleinberg-Shmoys ‘11][Sebo ‘13][Vygen ‘15][Gottschalk-Vygen ‘15][Sebo-van Zuylen ‘16][Traub-Vygen ‘18][Zenklusen ‘18][Traub-Vygen-Zenklusen ‘19]: Reducing path TSP to TSP

[Kisfaludi-Bak ‘20]: Quasi-polynomial hyperbolic TSP

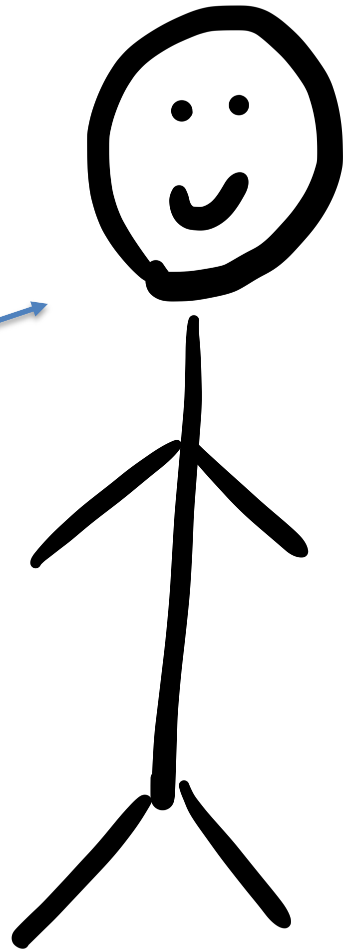
# Recent result

**Theorem** [Karlin-K-Oveis Gharan '20]: There is a randomized $1.5 - 10^{-36}$ approximation algorithm for metric TSP.




$\epsilon$ improvement in meters

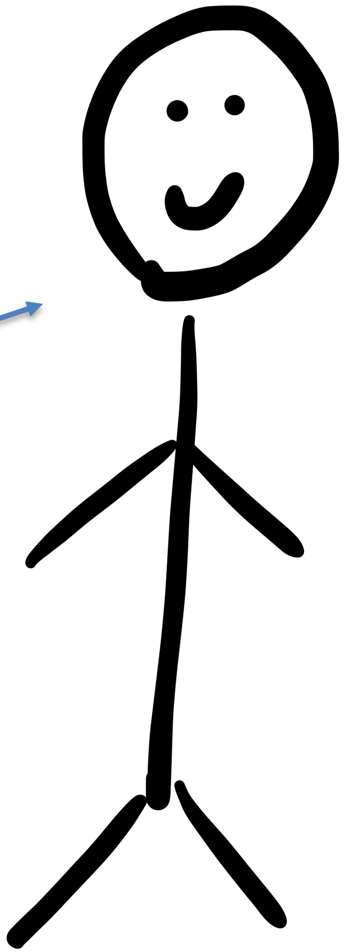
Me in Bonn 2021



Using randomness is cheating

András Sebő in Bonn

Me in Bonn 2021

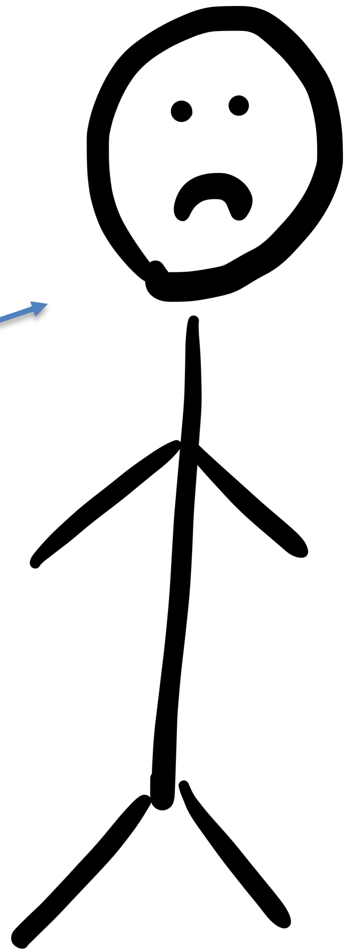


Using randomness is cheating

You haven’t beaten Christofides until you get a deterministic algorithm.

András Sebő in Bonn

Me in Bonn 2021



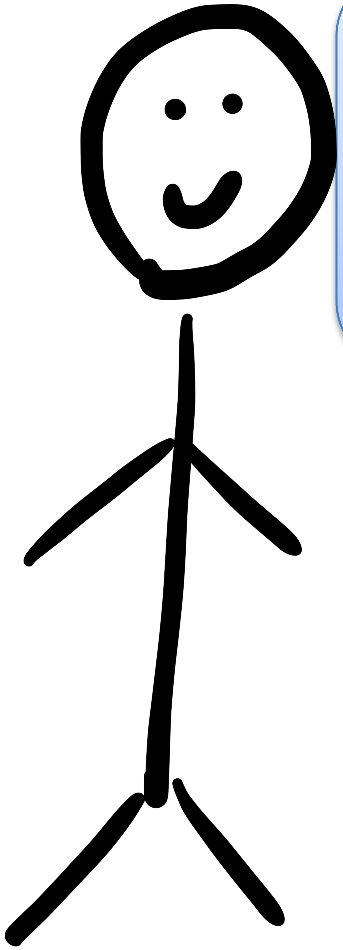
~~Using ra~~ ~~cheating~~

~~You~~ ~~beaten~~ ~~fides~~
~~until you~~
~~dete~~ ~~algo~~

(… It was definitely much nicer than this, maybe “it would be cool if you could derandomize it”)

András Sebő in Bonn

# This result



**Theorem:** There is a ~~randomized~~ deterministic $1.5 - 10^{-36}$ approximation algorithm for metric TSP.

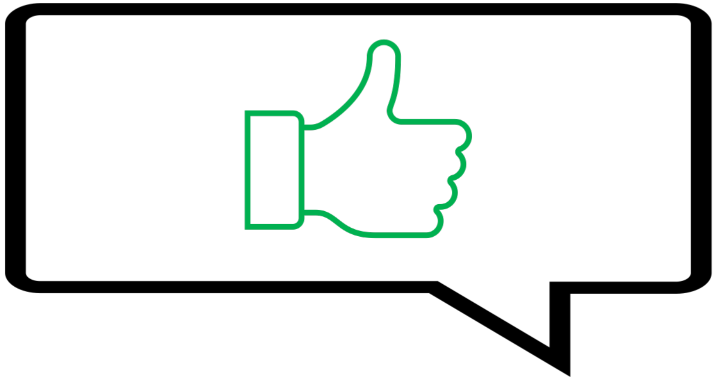


Photo credit: Bill Cook

# Outline

1. **Background and algorithm**

2. Computing $\mathbb{E}[c(T) \mid Set]$ with the matrix tree theorem

3. Defining $y(T)$ in the special “degree cut” case.

4. Computing $\mathbb{E}[\, c(y) \mid Set \,]$ in the degree cut case.

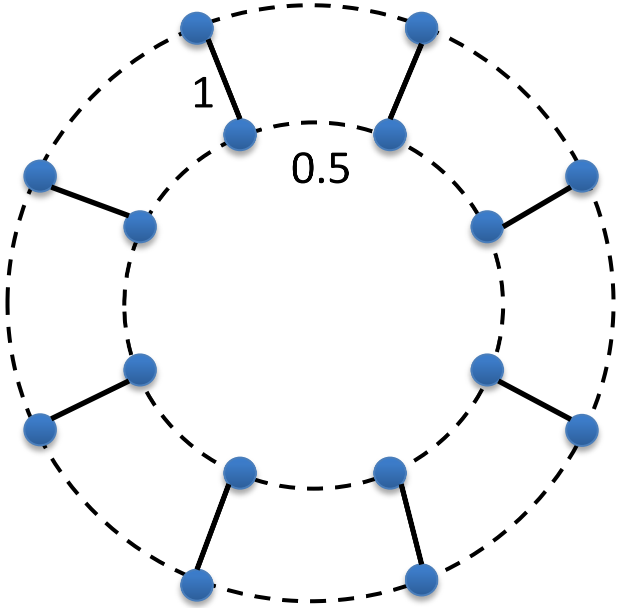
# Background #1: Linear programming relaxation

$$\min \sum_e c_e x_e$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \qquad \forall S \subset V$$

$$\sum_{e \in \delta(u)} x_e = 2 \qquad \forall u \in V$$

$$x_e \geq 0 \qquad \forall e$$




Subtour elimination LP/Held-Karp relaxation

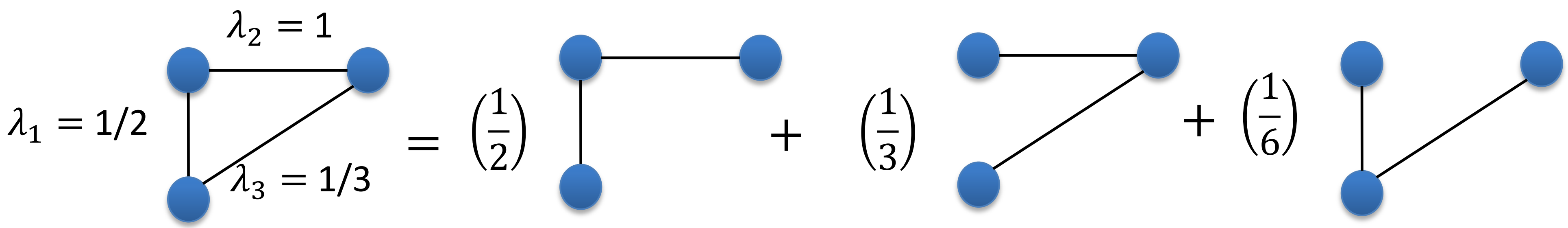
[Dantzig, Fulkerson, Johnson '54][Held and Karp '70]

# Background #2: $\lambda$-uniform spanning trees

For $\lambda : E \to \mathbb{R}_{\geq 0}$, the $\lambda$ −uniform spanning tree distribution sets:

$$\mathbb{P}[T] = \prod_{e \in T} \lambda_e = \lambda^T \text{ for all trees } T$$

Where we assume $\lambda$ is normalized such that $\sum_T \lambda^T = 1$.




[Asadpour, Goemans, Madry, Oveis Gharan, Saberi '10]: For any point $z$ in the spanning tree polytope, we can find a $\lambda$-uniform distribution in polynomial time (via a max entropy convex program) such that:

$$\forall e, \mathbb{P}_{T \sim \mu_\lambda}[e \in T] = z_e$$

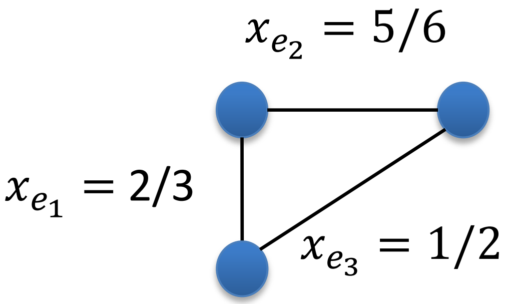
Up to exponentially small multiplicative error.

# Example

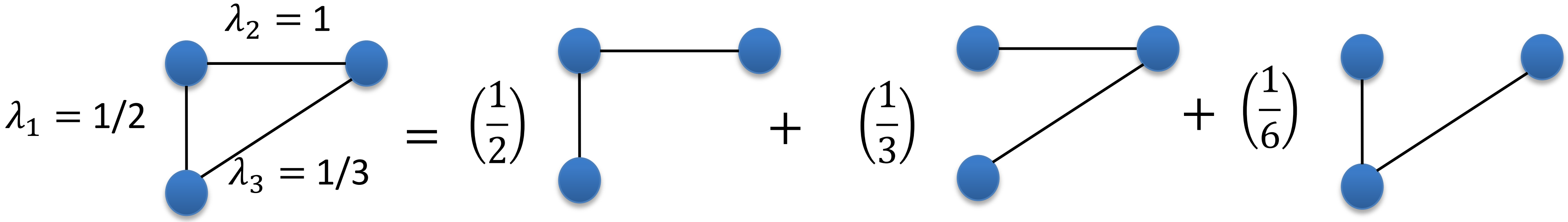
For $\lambda: E \to \mathbb{R}_+$, a $\lambda$ −uniform spanning tree distribution satisfies

$$\mathbb{P}_\lambda[T] = \prod_{e \in T} \lambda_e$$

Suppose we get this point in the spanning tree polytope

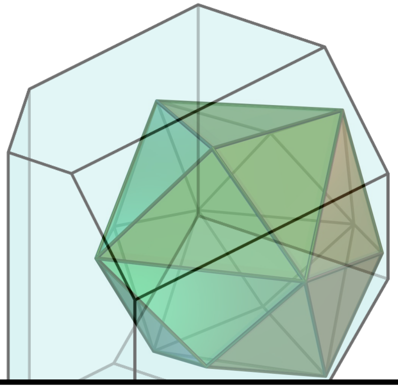



Then we will produce this vector $\lambda$ and thus a distribution over spanning trees



# Max entropy tree algorithm for TSP

Slight variant of [Oveis Gharan, Saberi, Singh '10]

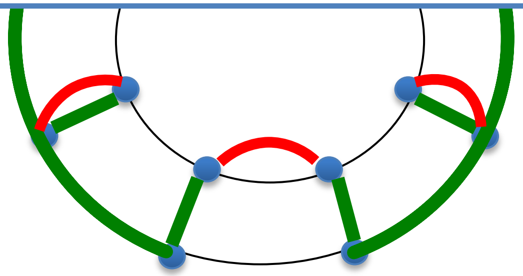


- Compute an LP solution $x$ to the subtour LP
- Find a $\lambda$-uniform distribution $\mu_\lambda$ with marginals $x$
- Sample $T \sim \mu_\lambda$ ⟹ $\boxed{\mathbb{E}[c(T)] = c(x) \leq OPT}$
- Add the minimum cost matching $M$ on the odd degree vertices of $T$

The subtour polytope is (almost) contained in the spanning tree polytope

*** Using properties of $\lambda$-uniform trees… ***

**Main Theorem** [KKO'20] **:** $\mathbb{E}[c(M)] \leq \left(\frac{1}{2} - \epsilon\right) OPT$ for some $\epsilon > 10^{-36}$.



# A Deterministic $\frac{3}{2} - \epsilon$ Approximation Algorithm for TSP

Obvious corollary!

**Main Theorem** [KKO'20] **:** $\mathbb{E}[c(T) + c(M)] \leq \left(\frac{3}{2} - \epsilon\right) OPT$ for some $\epsilon > 10^{-36}$.

- *Compute an LP solution $x$ to the subtour LP*
- *Find a $\lambda$-uniform distribution $\mu_\lambda$ with marginals $x$*
- For each $T$ in the support of $\mu_\lambda$, compute $c(T) + c(M)$.
- Output the tree that minimizes $c(T) + c(M)$.

**Issue:** There are exponentially many trees in the support of $\mu_\lambda$.

# A Deterministic $\frac{3}{2} - \epsilon$ Approximation Algorithm for TSP

Attempt #2

**Main Theorem** [KKO'20] **:** $\mathbb{E}[c(T) + c(M)] \leq \left(\frac{3}{2} - \epsilon\right) OPT$ for some $\epsilon > 10^{-36}$.

- *Compute an LP solution $x$ to the subtour LP*

**Issue:** how do we compute this quantity?

To sample the tree deterministically

- For each edge $e$:
  - If $\boxed{\mathbb{E}[\, c(T) + c(M) \mid Set, e \in T\,]} \leq \mathbb{E}[c(T) + c(M) \mid Set, e \notin T]$:
    - Let $Set = Set \cup \{e = 1\}$
  - Else, let $Set = Set \cup \{e = 0\}$.
- Let $T$ be the set of edges set to 1 in $Set$.
- *Add the minimum cost matching $M$ on the odd degree vertices of $T$*

# A short remark and the main theorem

**Issue:** how do we compute $\mathbb{E}[\, c(T) + c(M) \mid Set, e \in T\,]$?

Notice: the *proof* upper bounds this quantity.

Can we make the *proof* polytime?

**Main Theorem** [KKO'20] **:**

$$\mathbb{E}[c(M) + c(T)] \leq \left(\frac{3}{2} - \epsilon\right) OPT$$

Here the analysis used the optimal solution. Thus there was really no hope of derandomizing this computation.

**Main Theorem** [KKO'21] **:**

$$\mathbb{E}[c(M) + c(T)] \leq \left(\frac{3}{2} - \epsilon\right) c(x)$$

In this result, the analysis did not use the optimal solution. This bounded the integrality gap and gave hope for derandomization.

We show that the analysis in [KKO'21] can be made into a polynomial time algorithm.

**Main Theorem:** There exists a random variable $y\colon \mathcal{T} \rightarrow \mathbb{R}_{\geq 0}^{E}$ such that:

1. $\forall\, T \in \mathcal{T}$, $c(M) \leq c(y)$ ($M$ is min cost matching).
2. $\mathbb{E}[c(T) + c(y)] \leq \left(\frac{3}{2} - \epsilon\right) c(x)$.
3. For any setting $Set$ of edges in/out of the tree we can compute $\mathbb{E}[\, c(T) + c(y) \mid Set\,]$ in polynomial time.

i.e., $c(y)$ is a pessimistic estimator for $c(M)$ that we can efficiently compute and has cost below $\frac{x}{2}$

**Condition 3:** For any setting $Set$ of edges in/out of the tree we can compute $\mathbb{E}[\, c(T) + c(y) \mid Set \,]$ in polynomial time. (Need to show!)

**Shows that the algorithm can be implemented in polynomial time.**

To sample the tree deterministically

- *Compute an LP solution $x$ to the subtour LP*
- *Find a $\lambda$-uniform distribution $\mu_\lambda$ with marginals $x$*
- Initialize $Set = \emptyset$.
- For each edge $e$:
  - If $\mathbb{E}[\, c(T) + c(y) \mid Set, e \in T \,] \leq \mathbb{E}[c(T) + c(y) \mid Set, e \notin T]$:
    - Let $Set = Set \cup \{e = 1\}$
  - Else, let $Set = Set \cup \{e = 0\}$.
- Let $T$ be the set of edges set to 1 in $Set$.
- *Add the minimum cost matching $M$ on the odd degree vertices of $T$*

**Goal:** compute $\mathbb{E}[\, c(T) + c(y) \mid Set \,]$ for any possible setting of edges in/out of the tree $Set$.

# Outline

1. Background and algorithm
2. **Computing $\mathbb{E}[\boldsymbol{c(T)} \mid \boldsymbol{Set}]$ with the matrix tree theorem**
3. Defining $y$ in the special "degree cut" case.
4. Computing $\mathbb{E}[\, c(y) \mid Set \,]$ in the degree cut case.

# Key definition

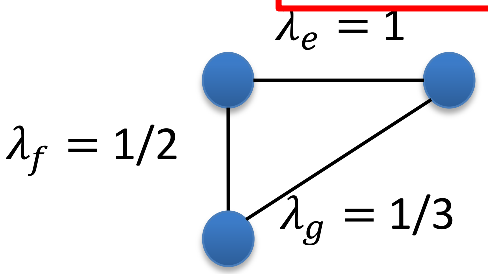
**Generating polynomial:** Let $\mu_\lambda$ be a $\lambda$-uniform distribution over spanning trees of a graph $G = (V, E)$. For each $e \in E$, define a variable $z_e$.

Then the generating polynomial of $\mu_\lambda$ is:

$$g_{\mu_\lambda}(\{z_e\}_{e\in E}) = \sum \mu(T) z^T = \sum \lambda^T z^T$$

Where we define $z^T =$

Why is this useful? It has exponentially many terms




$$\boxed{g_{\mu_\lambda}(\{z_e\}_{e\in E}) = \frac{1}{2} z_e z_f + \frac{1}{3} z_e z_g + \frac{1}{6} z_f z_g}$$

# The matrix tree theorem

[Kirchoff 1847]

**Matrix Tree Theorem:** Let $\mu_\lambda$ be a $\lambda$-uniform distribution over spanning trees on a graph $G = (V, E)$. Then,

$$g_{\mu_\lambda}(\{z_e\}_{e\in E}) = \sum_T \mu(T) z^T = \sum_T \lambda^T z^T$$
$$= \det\left(\sum_e \lambda_e z_e L_e + \frac{\mathbf{1}\mathbf{1}^T}{n^2}\right)$$

Where for an edge $e = (u, v)$,

$$L_e = (1_u - 1_v)(1_u - 1_v)^T$$

is the Laplacian of the edge $e$.

**Upshot:** we can compute the value of $g_{\mu_\lambda}(z)$ at any point $z \in \mathbb{C}^{|E|}$ in polynomial time.

Therefore, we can easily compute $\mathbb{E}[c(T) \mid Set]$.

It remains to compute $\mathbb{E}[\, c(y) \mid Set \,]$.

**Question 1**: Given a $\lambda$-uniform distribution $\mu_\lambda$ and an oracle to compute $g_{\mu_\lambda}$ how do we compute $\mathbb{P}_{T\sim\mu_\lambda}[f \in T]$ for an edge $f$?

**Answer:** $\mathbb{P}[f \in T] = 1 - \mathbb{P}[f \notin T] = 1 - g_{\mu_\lambda}(z)$ where $z_f = 0$ and $z_e = 1$ for $e \neq f$, as:

$$g_{\mu_\lambda}(z) = \sum_{T: f \notin T} \lambda^T$$

**Question 2**: How do we compute $\mathbb{P}[f \in T \mid Set]$ for an edge $f$?

**Answer:** First, contract all edges set to 1 and delete all edges set to 0. We have a resulting $\lambda'$-uniform distribution $\mu_\lambda$, on a graph $G'$. First, renormalize $\lambda'$. Then apply the above.

**Goal:** compute $\mathbb{E}[\, c(T) + c(y) \mid Set\,]$ for any possible setting of edges in/out of the tree $Set$.

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# Interlude: The $Odd(T)$-Join polyhedron $P_{O(T)}$

[Edmonds and Johnson '73]

$$\min \sum_{u,v} c_e y_e$$

$$\sum_{e \in \delta(S)} y_e \geq 1 \qquad \forall S \; |\delta(S) \cap E(T)| \text{ odd}$$

$$y_e \geq 0 \qquad \forall e$$

Equivalent $\longleftrightarrow$

$\forall S$ containing an odd number of odd vertices in the tree

Has an integrality gap of 1.

So, if $y$ is in the $Odd(T)$-Join polyhedron $P_{O(T)}$, then $c(M) \leq c(y)$. We will ensure this, implying $c(y)$ is a pessimistic estimator.

# Degree cut case

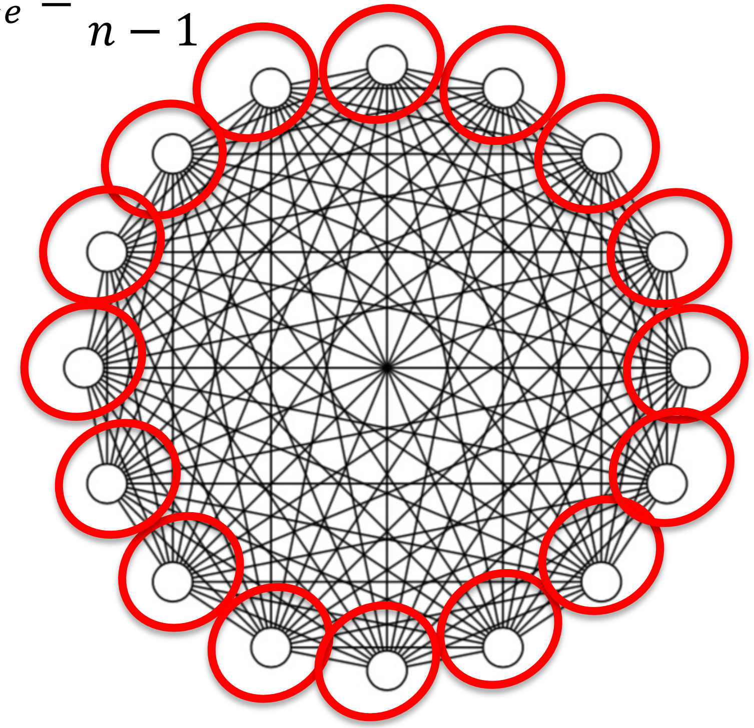
Subtour LP constraints

$$\sum_{e \in \delta(S)} x_e \geq 2 \qquad \forall S \subset V$$

$$\sum_{e \in \delta(u)} x_e = 2 \qquad \forall u \in V$$

$$x_e \geq 0 \qquad \forall e$$

$$x_e = \frac{2}{n-1}$$



$Odd(T)$-Join constraints

$$\sum_{e \in \delta(S)} y_e \geq 1 \qquad \forall S \; |\delta(S) \cap E(T)| \text{ odd}$$

$$y_e \geq 0 \qquad \forall e$$

Suppose that the only (really) small cuts in the LP solution $x$ are the vertices.

In other words, suppose all cuts $S \subseteq V$ with $2 \leq |S| \leq n-2$ have

$$\sum_{e \in \delta(S)} x_e \geq 2 + \eta$$

for some absolute constant $\eta > 0$.

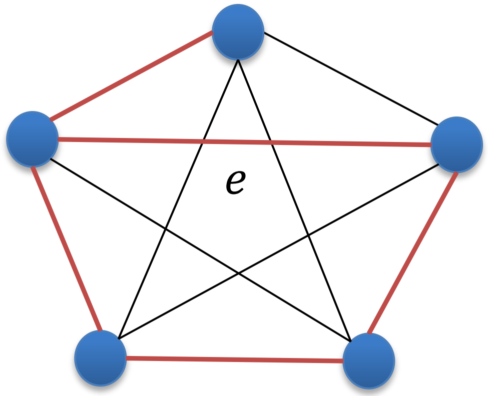
# An estimator $y$ for the degree cut case

$Odd(T)$-Join constraints

$$\sum_{e \in \delta(S)} y_e \geq 1 \qquad \forall S \ |\delta(S) \cap E(T)| \text{ odd}$$

$$y_e \geq 0 \qquad \forall e$$

+ We assume all non-vertex cuts have at least $2 + \eta$ mass going across in $x$.




$$y_e = \frac{x_e}{2+\eta}$$

For an edge $e = (u, v)$, we let:

$$y_e = \begin{cases} \frac{x_e}{2+\eta} & \text{if } u \text{ and } v \text{ both have even degree in } T \\ \frac{x_e}{2} & \text{otherwise} \end{cases}$$

# Claim: $y(T)$ is in $P_{O(T)}$

For an edge $e = (u, v)$, we let:

$$y(T)_e = \begin{cases} \frac{x_e}{2+\eta} & \text{if } u \text{ and } v \text{ both have even degree in } T \\ \frac{x_e}{2} & \text{otherwise} \end{cases}$$

**Proof:** For any cut $2 \leq |S| \leq n-2$, $x\big(\delta(S)\big) \geq 2+\eta$ (by assumption). Since $y_e \geq \frac{x_e}{2+\eta}$,

$$\sum_{e \in \delta(S)} y_e \geq \frac{1}{2+\eta} \sum_{e \in \delta(S)} x_e \geq \frac{1}{2+\eta} \cdot (2+\eta) = 1$$

For the vertices: if a vertex $v$ is even, there is no constraint. If $v$ is odd, then all $e \sim v$ have $y_e = \frac{1}{2} x_e$, so since $\sum_{e \in \delta(v)} x_e = 2$ we have $\sum_{e \in \delta(v)} y_e = 1$.

$Odd(T)$-Join constraints ($P_{O(T)}$)

$$\sum_{e \in \delta(S)} y_e \geq 1 \qquad \forall S\ |\delta(S) \cap E(T)| \text{ odd}$$

$$y_e \geq 0 \qquad \forall e$$

So, $y$ is a simple pessimistic estimator for the cost of the min-cost matching and it has $\mathbb{E}[c(y)] \leq \left(\frac{1}{2} - \epsilon\right) c(x)$ at the beginning.

**Now we'll show how to compute $\mathbb{E}[\,c(y)\mid Set\,]$ for this simple $y$.**

$$\mathbb{E}[y_e] = \underbrace{\frac{x_e}{2}}_{\text{Some endpoint of } e \text{ is odd}}(1 - \mathbb{P}[u, v \text{ even degree in } T]) + \mathbb{P}[u, v \text{ even degree in } T]\overbrace{\frac{x_e}{2+\eta}}^{\text{Both endpoints of } e \text{ are even}}$$

$$= \left(\frac{1}{2} - \frac{\eta}{4+2\eta}\mathbb{P}[u, v \text{ even degree in } T]\right)x_e \leq \left(\frac{1}{2} - \frac{\eta p}{4+2\eta}\right)x_e$$

# Outline

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4. **Computing $\mathbb{E}[\, c(y) \mid Set \,]$ in the degree cut case.**

- *For all trees $T \in \mathcal{T}$, $c(M) \leq c(y)$ where $M$ is the minimum cost matching on the odd vertices of $T$.* ☑
- $\mathbb{E}[c(T) + c(y)] \leq \left(\frac{3}{2} - \epsilon\right) c(x)$. ☑
- For any setting $Set$ of edges in/out of the tree we can compute $\mathbb{E}[\, c(T) + c(y) \mid Set \,]$ in polynomial time.

Let $e = (u, v)$. Then,

$$\mathbb{E}[y_e] = \left(\frac{1}{2} - \frac{\eta}{4 + 2\eta}\mathbb{P}[u, v \text{ even degree in } T]\right) x_e$$

Therefore,

$$\mathbb{E}[y_e \mid Set] = \left(\frac{1}{2} - \frac{\eta}{4 + 2\eta}\mathbb{P}[u, v \text{ even degree in } T \mid Set]\right) x_e$$

So, to compute $\mathbb{E}[c(y)]$, it is enough to compute $\mathbb{P}[u, v \text{ even degree in } T \mid Set]$ for all $e = (u, v)$.

This is straightforward using the generating polynomial $g_{\mu_\lambda}$

# Computing $\mathbb{P}[u, v \text{ even degree in } T \mid Set]$

**Observation**: It is easy to condition on $Set$.

We've seen this before! Contract all edges set to 1, delete all edges set to 0, and renormalize.
So, all we have to do is compute $\mathbb{P}[u, v \text{ even degree in } T]$ for a $\lambda$-uniform distribution.

**Warmup**: Remember we can compute $g_{\mu_\lambda}(\{z_e\}_{e \in E}) = \sum_T \mu(T) z^T = \sum_T \lambda^T z^T$ at any point $z$.
If I give you a set $F \subseteq E$, how do we compute $\mathbb{P}[|F \cap T| \text{ even}]$?

So, $\mathbb{P}[|F \cap T| \text{ even}] = \frac{1}{2}(g_{\mu_\lambda}(z) + 1)$

# Computing $\mathbb{P}[u, v \text{ even degree in } T \mid Set]$

**Lemma**: We can compute $\mathbb{P}[|A \cap T|, |B \cap T| \text{ even} \mid Set]$ for any sets of (not necessarily disjoint) edges $A, B \subseteq E$.

From the previous slide, we can drop $Set$. Now observe:

$$\mathbb{I}[|A \cap T|, |B \cap T| \text{ even}] = \frac{1}{4}\left(1 + (-1)^{|A \cap T|} + (-1)^{|B \cap T|} + (-1)^{\left|\left((A \setminus B) \cup (B \setminus A)\right) \cap T\right|}\right)$$

Recall for $F \subseteq E$ and the point $z^F$ where $z_e^F = 1$ for $e \notin F$, $z_e^F = -1$ for $e \in F$, we have

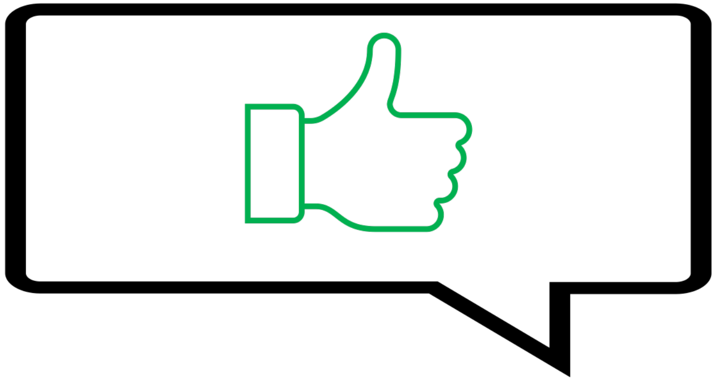
$$g_{\mu_\lambda}(z) = \sum_T \mu(T)(-1)^{|F \cap T|} = \mathbb{E}[(-1)^{|F \cap T|}]$$

**Corollary**: We can compute $\mathbb{P}[u, v \text{ even degree in } T \mid Set]$.

So, we have a deterministic algorithm in the degree cut case.

# Using the actual (complicated) $y$ from [KKO’21], after some more work…

**Theorem:** There is a deterministic ~~randomized~~ $1.5 - 10^{-36}$ approximation algorithm for metric (path) TSP.



Traub, Vygen, Zenklusen in 2019

# Key Derandomization Lemma

**Lemma**: For any sets of (not necessarily disjoint) edges $A_1, \ldots, A_k \subseteq E$, and any $\sigma_1, \ldots, \sigma_k \in \mathbb{F}_{r_1} \times \cdots \times \mathbb{F}_{r_k}$ and any $\lambda$-uniform distribution $\mu_\lambda$, we can compute

$$\mathbb{P}_{T \sim \mu_\lambda}[|A_i \cap T| = \sigma_i \pmod{r_i} \text{ in } T\ \forall 1 \leq i \leq k \mid Set]$$

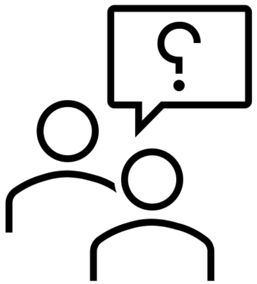
in time polynomial in $r_1 r_2 \ldots r_k$ (so, polynomial for any constant $k$).

Note: we only need this for $r_i \in \{2, n-1\}$, i.e. we are only interested in parity and cardinality.

# Open questions

**Open questions:**

- Can we directly compute $\mathbb{E}[c(T) + c(M) \mid Set]$ deterministically?
- Are there tree distributions with *polynomial sized support* that beat 3/2?
  (True for the degree cut case! [Hadaddan-Newman ’19])
- Can we improve the analysis for this algorithm?



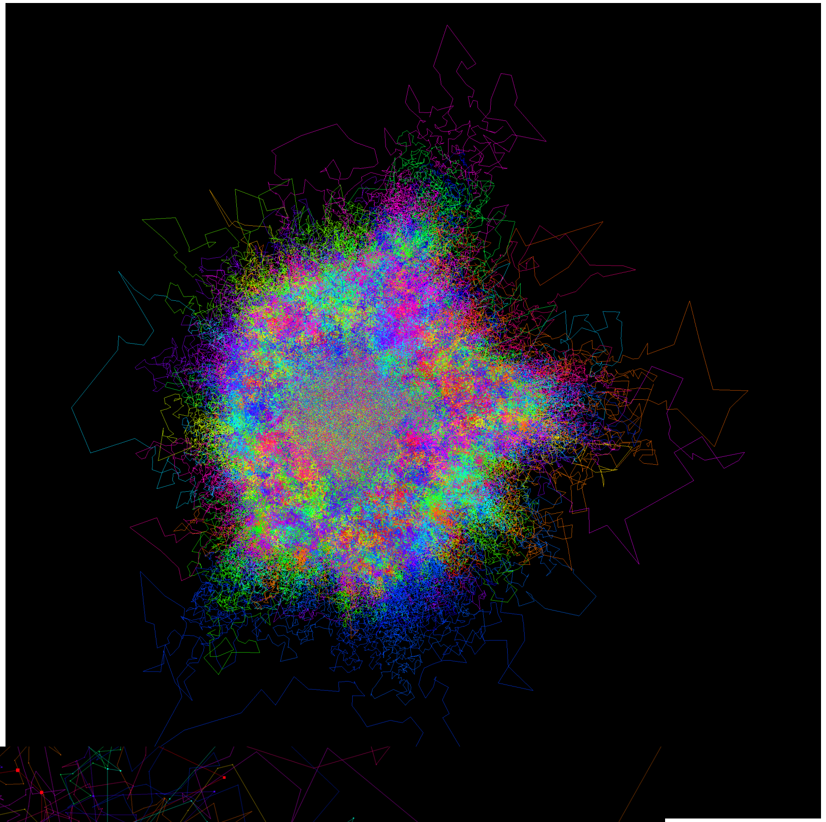
# Thank you!

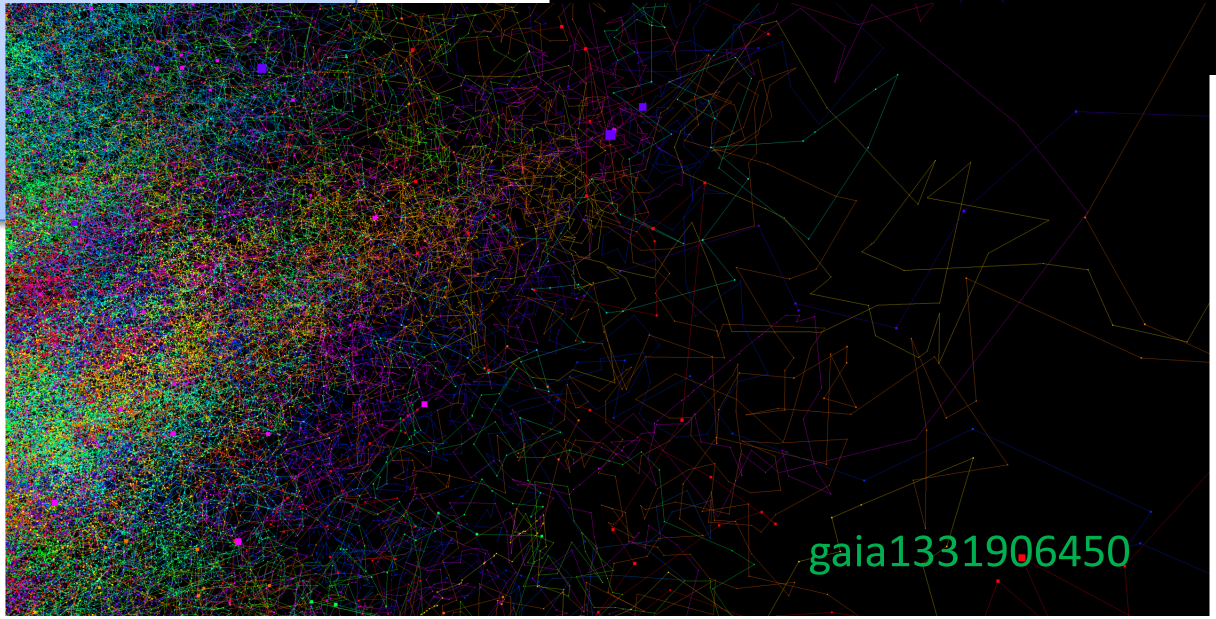
TSPortrait of Dantzig by Robert Bosch, 2006

# Previous result

**Theorem** [Karlin-K-Oveis Gharan '20]: There is a randomized $1.5 - 10^{-36}$ approximation algorithm for metric TSP.

On this tour we would gain the width of an atom!





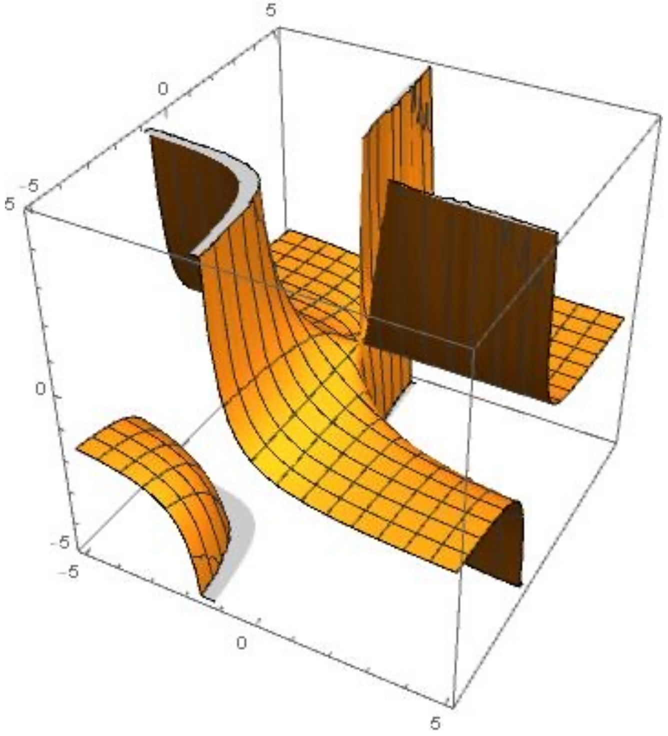
First, we
uniform
distribut
Rayleigh

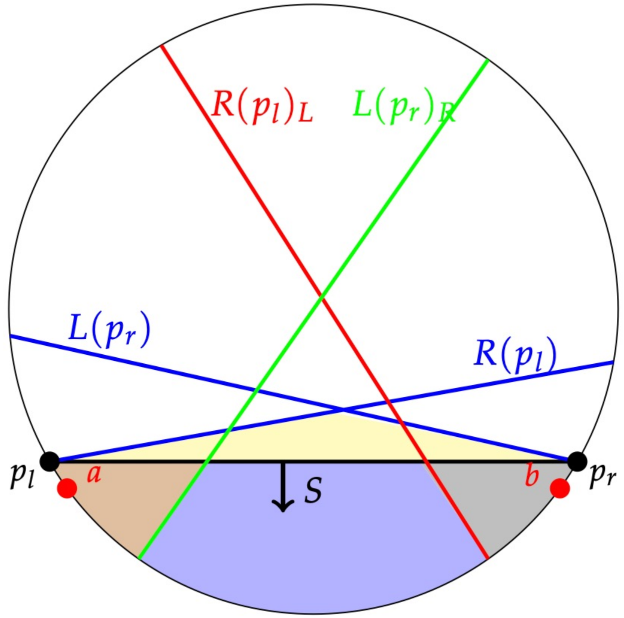
for

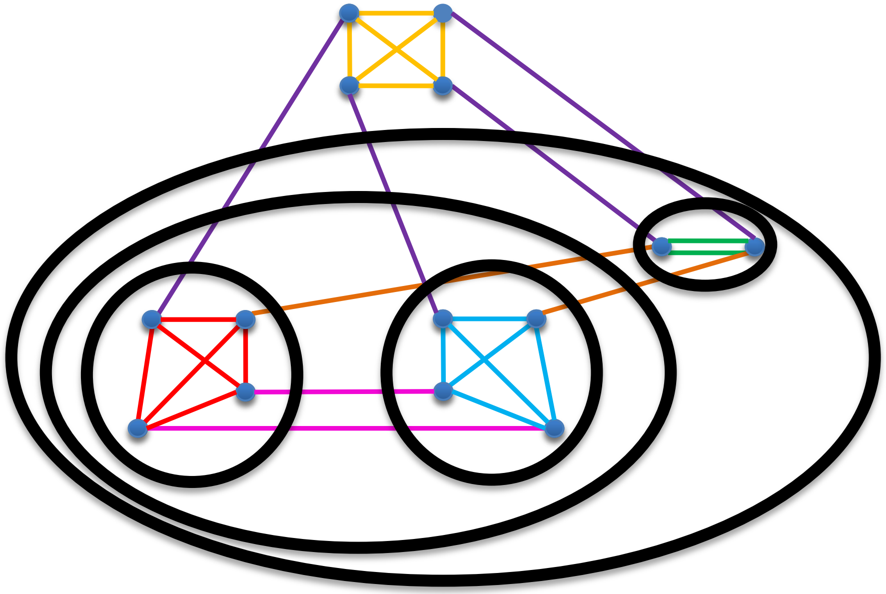
We derandomize:

1. The probabilistic computations
2. The uncrossing operations
3. The construction of $y$ for the laminar family

cuts





**Generating polynomial:** Let $\mu: 2^E \to \mathbb{R}_{\geq 0}$ be a distribution over ground set $E$. For each $e \in E$, define a variable $z_e$. Then the generating polynomial of $\mu$ is defined as:

$$g_\mu(\{z_e\}_{e\in E}) = \sum_{S\in 2^E} \mu(S) z^S$$

Where we define $z^S = \prod_{e\in S} z_e$.

**Example:** For a $\lambda$-uniform distribution of spanning trees over a graph $G = (V, E)$, the generating polynomial is:

$$g_{\mu_\lambda}(\{z_e\}_{e\in E}) = \sum_{T\in\mathcal{T}} \mu(T) z^T = \sum_{T\in\mathcal{T}} \lambda^T z^T$$