Eidgenössische Technische Hochschule Zürich
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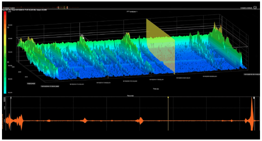
Department of Mathematics

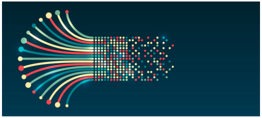
Constrained Sparse Approximation Over the Cube

Sabrina Bruckmeier Christoph Hunkenschröder Robert Weismantel

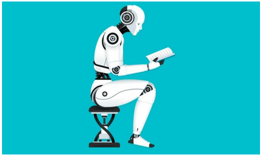
Sparse Approximation is the problem of identifying a subset that most accurately models an observation.
(Sebastian Ament and Carla Gomes)



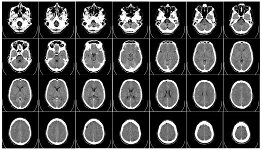
(Signal Recovery)



(Pattern Recognition)



(Machine Learning)



(Computed Tomography)

(Portfolio Selection)

Mathematical Description.

- Given:
  - Matrix $A \in \mathbb{Z}^{m \times n}$,
  - Vector $b \in \mathbb{Z}^m$,
  - Integer $\sigma \in [n]$.
- Task:

$$\min\{\|b - Ax\|_2 : x \in [0,1]^n, \|x\|_0 \leq \sigma\}. \qquad (P_0)$$

- In General: NP-Hard [1]
- Idea: Let's fix m.

[1] Tropp, J.A. (2004). "Greed is good: Algorithmic results for sparse approximation". IEEE Transactions on Information Theory. 50 (10): 2231–2242.

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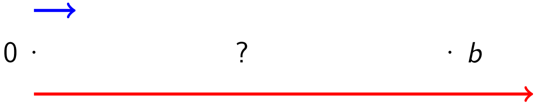
$$\min\{\|b - Ax\|_2 : x \in [0,1]^n, \|x\|_0 \leq \sigma\}.$$

- $m = 1$



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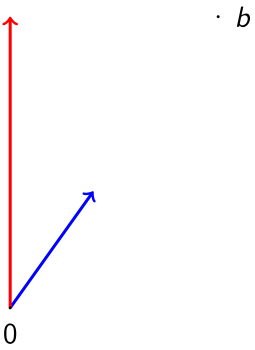
- $m = 1$

Or not?

- $m = 2$



Theorem

(P_0) is NP-hard, even if $m = 2$.

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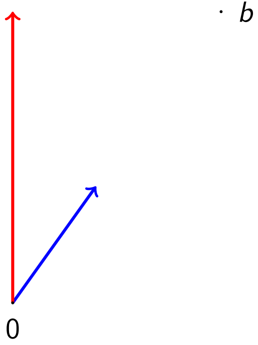

Theorem

(P_0) *is NP-hard, even if* $m = 2$.

Assumptions

- $\|A\|_\infty$ bounded
- m bounded
- $m \ll n$

$$A = \begin{bmatrix} \star & \star & \star & \star & \dots & \star & \star & \star \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \star & \star & \star & \star & \dots & \star & \star & \star \end{bmatrix}$$

ℓ_1-Relaxation.

$$\min\{\|b - Ax\|_2 : x \in [0,1]^n, \|x\|_0 \le \sigma\} \qquad (P_0)$$

$$\min\{\|b - Ax\|_2 : x \in [0,1]^n, \|x\|_1 \le \sigma\} \qquad (P_1)$$

So far

Algorithms

- need A to satisfy (hard to verify) properties.
- depend highly on the input.
- succeed only with a certain probability.

Our contribution

Independent of A we give

- Probabilistic analysis for random targets b.
- Proximity result between (P_0) and (P_1).
- Deterministic algorithm polynomial in n provided m and $\|A\|_\infty$ constant.

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Few fractional entries for (P_0).

- Feasible vector x with support S.
- Set of (P_0)-feasible points with same objective value:

$$P_S(x) := \{y \in \mathbb{R}^{|S|} : Ax = A_S y, 0 \leq y \leq 1\}.$$

- $P_S(x)$:
  - polyhedron,
  - non-empty,
  - has at least one vertex v.
- At v at least $|S| - m$ inequalities of the form $0 \leq y \leq 1$ are tight.

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There exists a solution of (P_0) that has at most m fractional entries.

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Section 1

The ℓ_1-relaxation for Random Targets b.

Which vectors b are "easy" target vectors?

- Set of points representable with the ℓ_1-relaxation

$$Q := \{Ax \in \mathbb{R}^m : x \in [0,1]^n, \|x\|_1 \leq \sigma\}.$$

- If b is "deep" inside Q, then (P_0) is easy.

Theorem

If $b \in \frac{\sigma-m+1}{\sigma} Q$, then an optimal solution of (P_1) solves (P_0).

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Proof (Sketch).

- Polyhedron $\{x \in [0,1]^n : Ax = b, \|x\|_1 \leq \sigma - m + 1\}$.
- Vertex v has at most $\sigma - m + 1$ integral non-zero entries.
- v has at most m fractional entries.
- $\|v\|_0 \leq \sigma$.
- v is optimal for (P_0).

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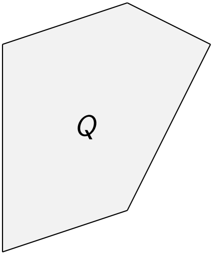

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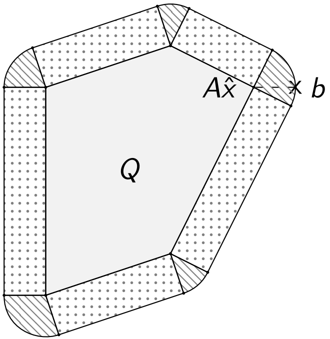


Figure: The sampling of the vector b from $Q + \lambda B$

Which vectors b are "easy" target vectors?

- If b is far outside of Q, then (P_0) is easy with high probability.

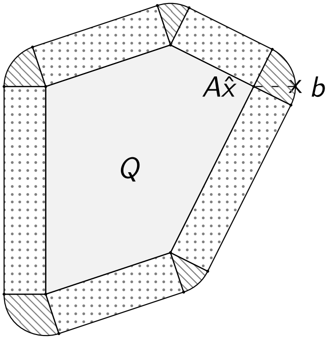


Figure: The sampling of the vector b from $Q + \lambda B$

Which vectors b are "easy" target vectors?

Theorem

If b is sampled uniformly at random from the convex set $Q + \lambda B$, then with probability at least

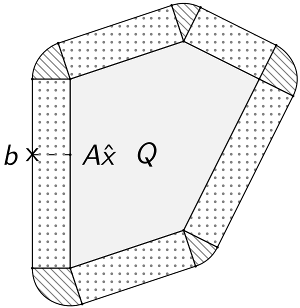
$$\rho = \left(\frac{\lambda}{\lambda + \sigma m \|A\|_\infty} \right)^m$$

there exists an optimal solution of (P_1) that is optimal for (P_0).

- Example: If $\lambda = 2m^2\sigma\|A\|_\infty$, then $\rho \geq \frac{1}{2}$ by Bernoulli's inequality.

Which vectors b are "easy" target vectors?

- Conversely, if b is close to the boundary of Q, then the probability that an optimal solution of (P_1) solves (P_0) is almost 0.



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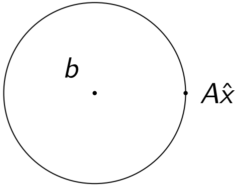
Section 2

Proximity between (P_1) and (P_0).

Separation Lemma.

- $\hat{x}$ optimal solution of (P_1).



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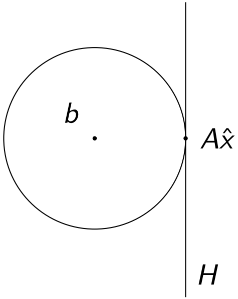

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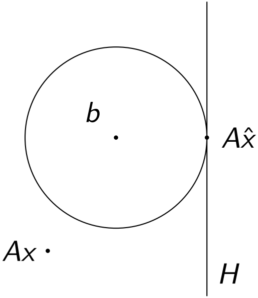

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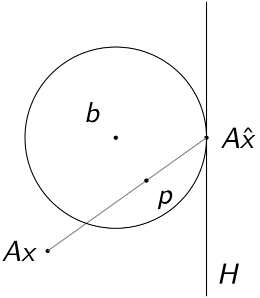

Lemma

H separates b from all vectors Ax with x feasible for (P_0).

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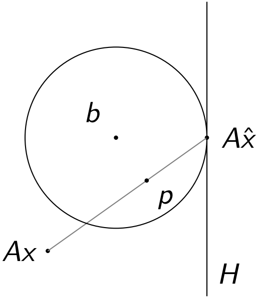

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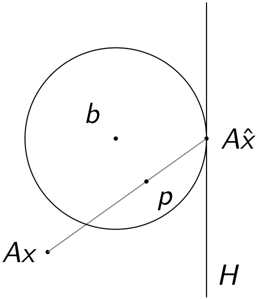
Perturbation Lemma.



Lemma

If we perturb $\hat{x}$ along the fractional entries, we will remain in H.

Proximity Result.

Theorem

For an optimal solution $x^\star$ to (P_0) we have

$$\|Ax^\star - A\hat{x}\|_2 \leq 2m^{3/2}\|A\|_\infty.$$

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Proof (Sketch)

- Reduce support of $\hat{x}$ by "filling up" the fractional entries $\rightarrow y$.
- y is feasible for (P_0).
- By the Perturbation Lemma $Ay \in H$.
- Separation Lemma + Standard Linear Algebra yields the result.

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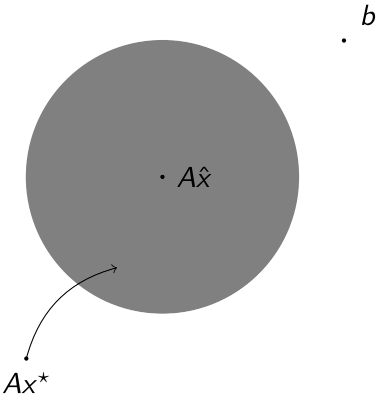
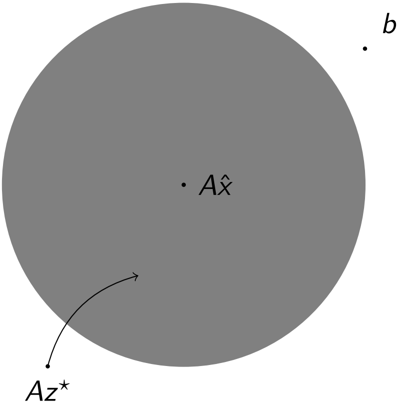
Section 3

A Deterministic Algorithm.


b
$A\hat{x}$
$Ax^\star$

Main Idea: Decompose $x^\star = z^\star + f^\star$.



1. Guess support $\text{supp}(f^\star)$ of fractional entries.

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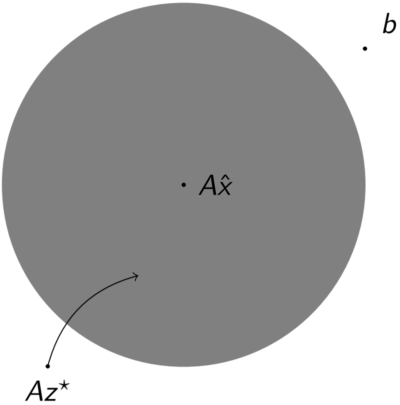

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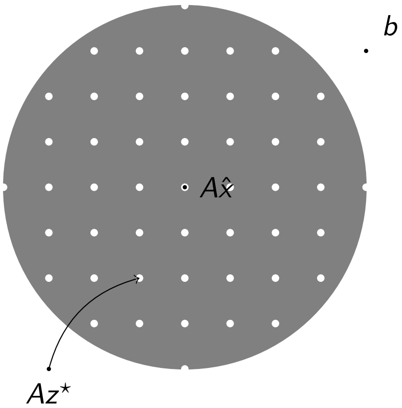

2. Establish candidate set $Z^\star$ for $z^\star$.

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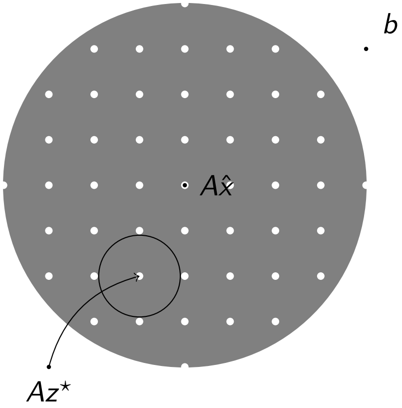

3. Match each $z \in Z^\star$ with its fractional part f.

Arithmetic Cost

- Guess support $\text{supp}(f^\star)$ of fractional entries.
  - $\text{supp}(f^\star) \leq m$.
  - Minimal index set of fractional entries uses distinct columns of A.
  - There are at most $(2\|A\|_\infty + 1)^m$ distinct columns.

Lemma

There are at most $(2\|A\|_\infty + 1)^{m^2}$ potentially different index sets $\text{supp}(f^\star)$.

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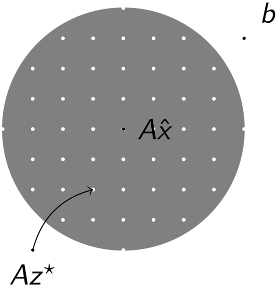


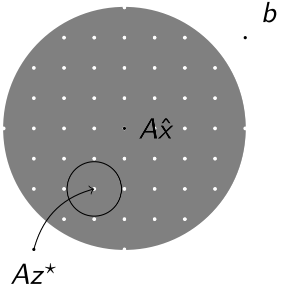
Theorem

Compute $Z^\star$ by solving at most $\mathcal{O}(m^{\frac{3}{2}}\|A\|_\infty)^m$ LIPs

$$A_{\setminus f^\star} y = b',$$
$$\sum_{i=1}^{n-m} y_i \leq \sigma - m,$$
$$y \in \{0,1\}^{n-m}.$$

Arithmetic Cost

- Match each $z \in Z^\star$ with its fractional part f.



Theorem

A solution of

$$\min\{\|(b - A_{\setminus f^\star} z^\star) - A_{f^\star} g\|_2 : \; g \in [0,1]^m\}$$

can be computed in $\mathcal{O}(3^m m^3)$ *arithmetic operations.*

Arithmetic Cost.

Theorem

The arithmetic cost of finding an optimal solution to (P_0) is

$$(m\|A\|_\infty)^{\mathcal{O}(m^2)} \cdot poly(n, \ln(\|b\|_1)).$$