FWF
Der Wissenschaftsfonds.

Lasse Wulf*

# A linear time algorithm for linearizing quadratic and higher-order shortest path problems

Joint work with Eranda Çela*, Bettina Klinz*, Stefan Lendl**, Gerhard Woeginger†

*Graz University of Technology
**University of Graz
†RWTH Aachen. Deceased April 2022

IPCO, June 23rd, 2023

Linear problem:

$$\min_{x \in X} cx$$

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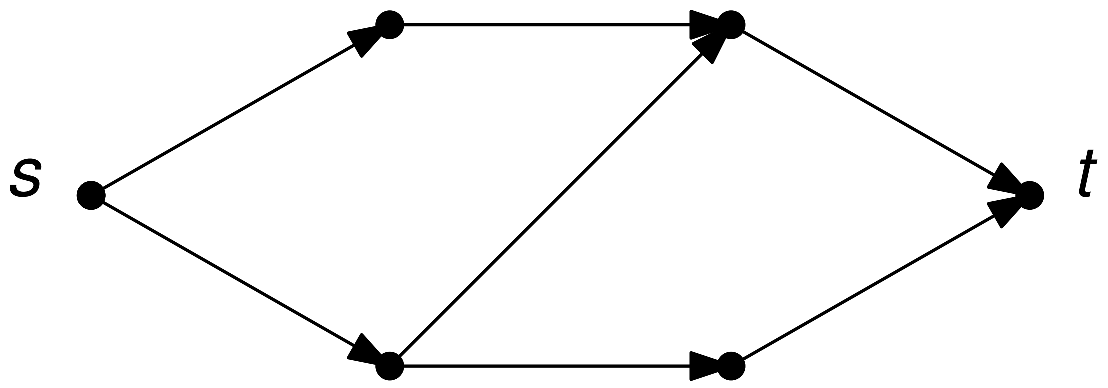
$$\min_{x \in X} cx$$

Quadratic problem:

$$\min_{x \in X} x^t Q x$$

# Shortest Path Problem (SPP)

$G = (V, E)$



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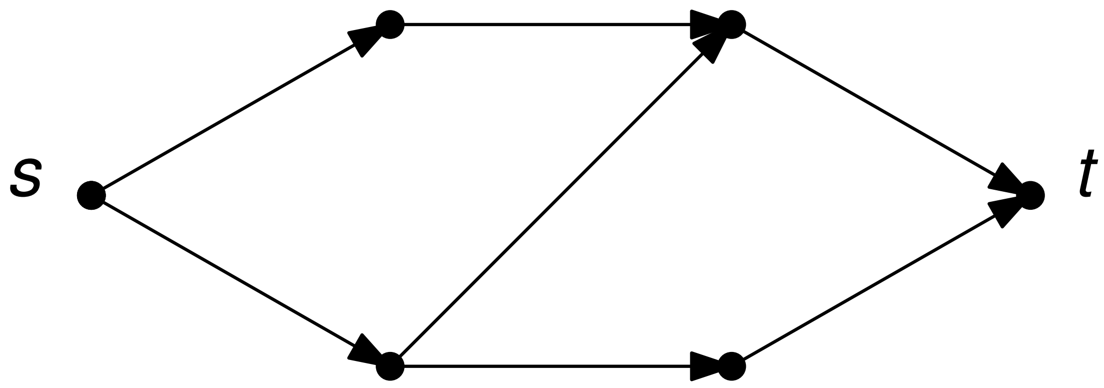

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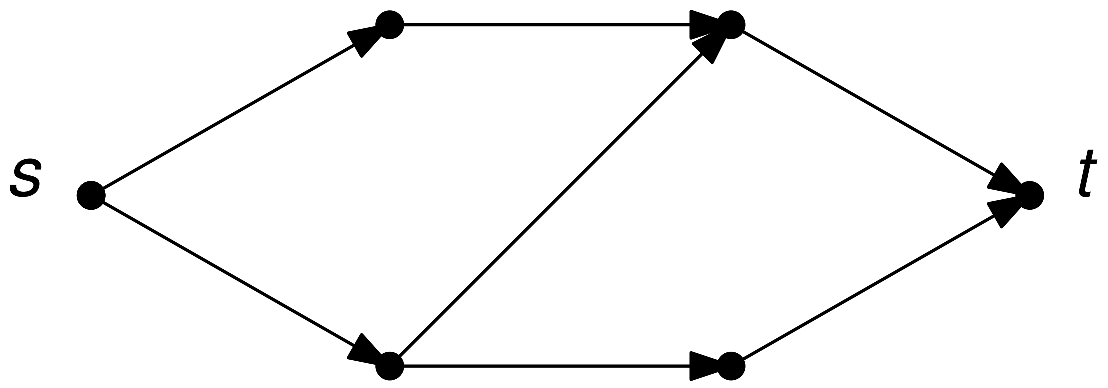
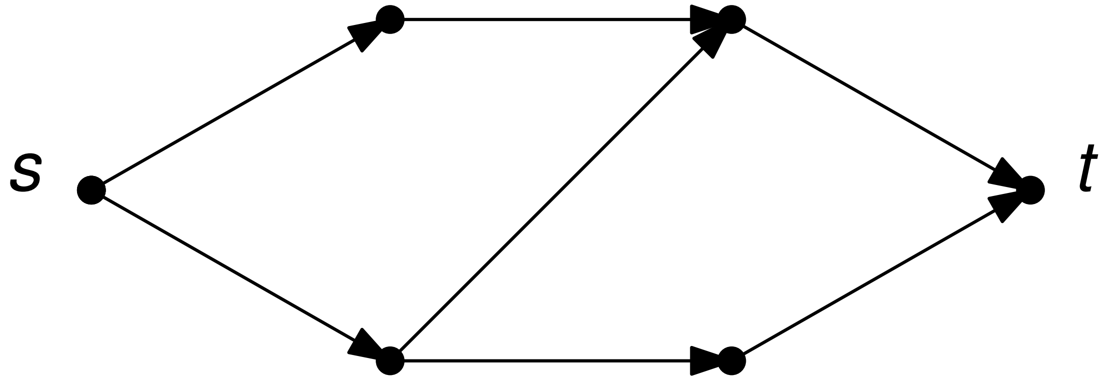


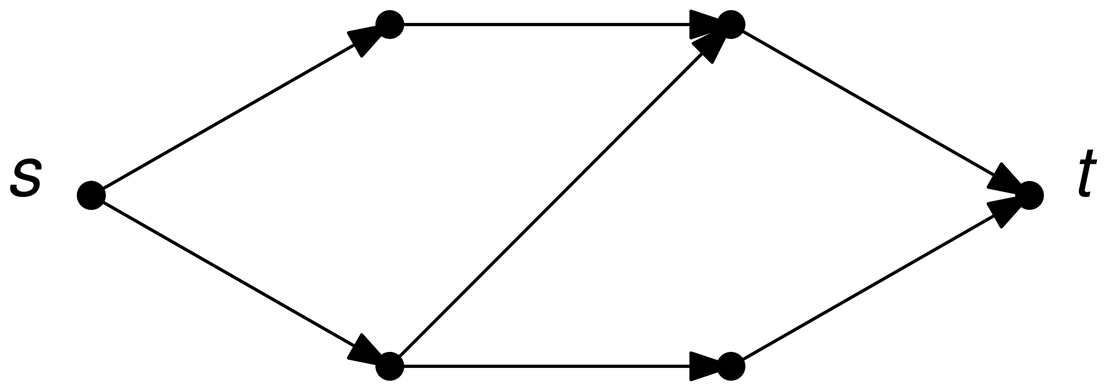

$$SPP(P, c) = \sum_{e \in P} c(e)$$

# Quadratic Shortest Path Problem (QSPP)



# Quadratic Shortest Path Problem (QSPP)

$$q : \{B \subseteq E : |B| \leq 2\} \to \mathbb{R}$$



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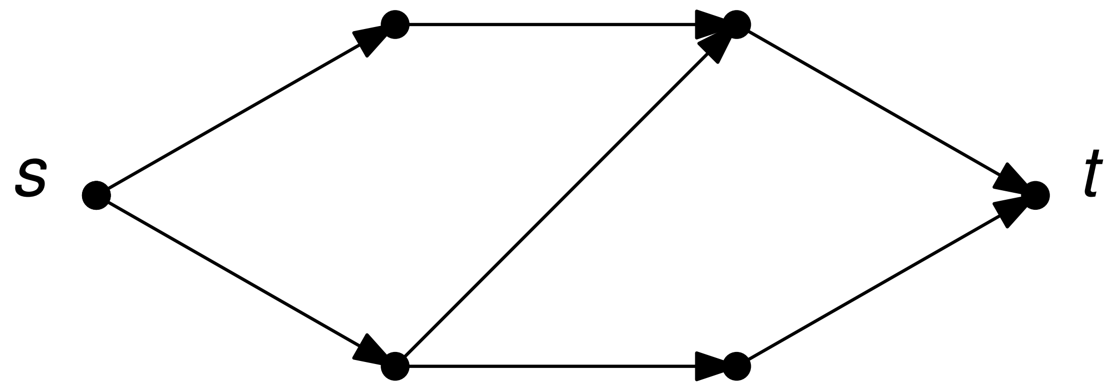


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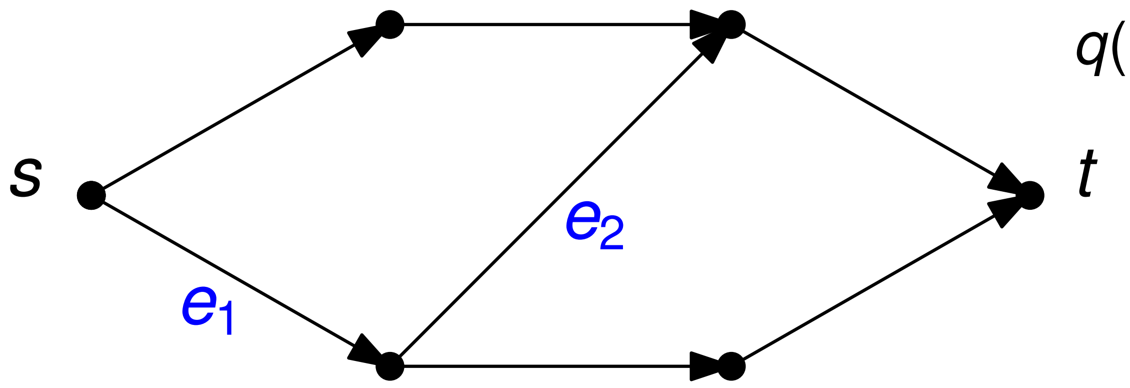


$q(\{e_1, e_2\}) = 3$
$q(B) = 0$ otherwise

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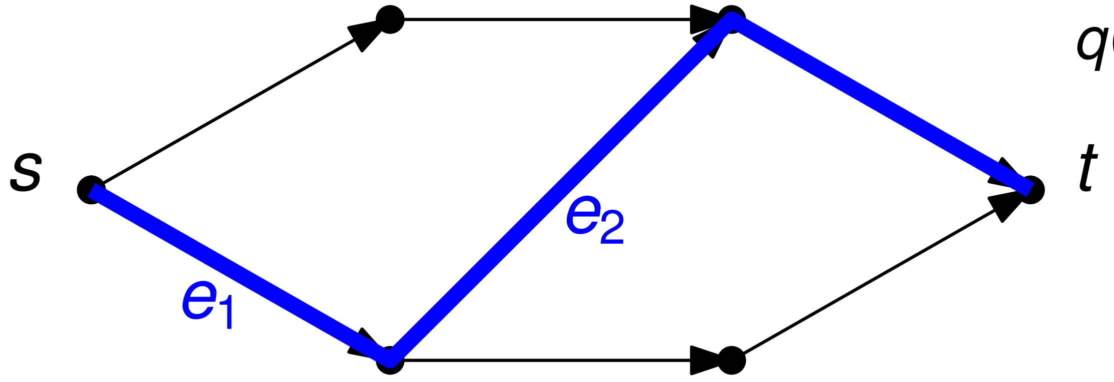


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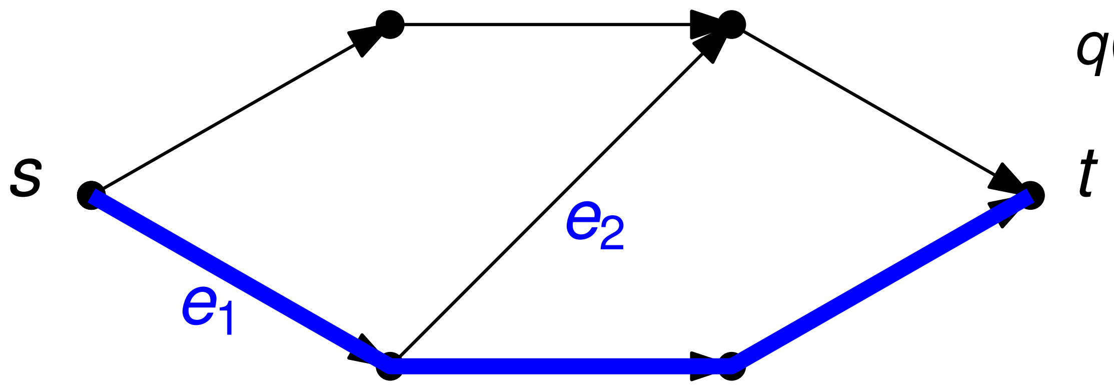


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# Applications

- stochastic and time-dependent route planing
- network design
- [Rostami et al. ’18]: overview of applications

QSPP

NP-complete :(

very slow :(

SPP

very fast :)

# Linearizations

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 $\Rightarrow$  ?

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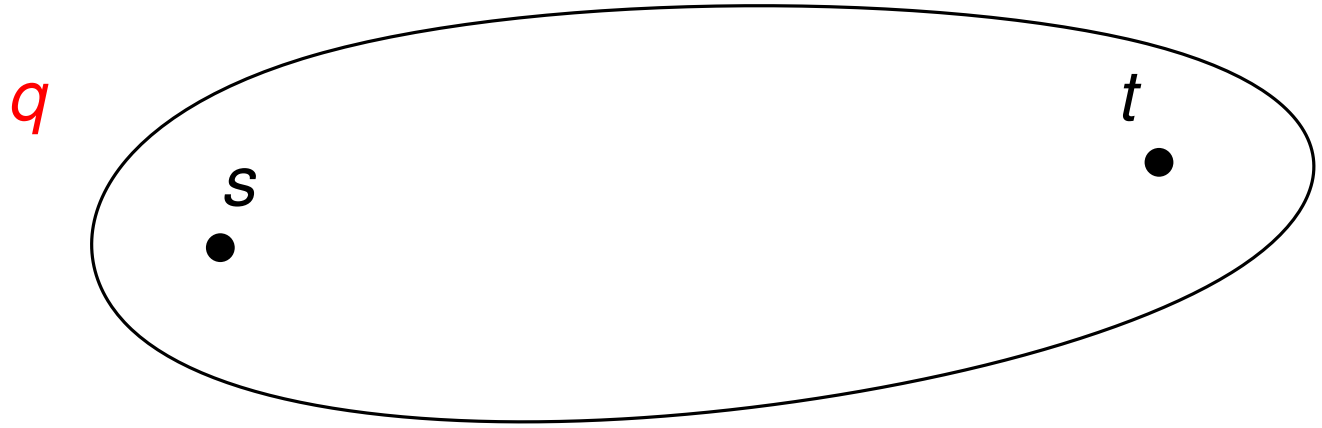
A QSPP instance is linearizable, if

$$\exists c : E \to \mathbb{R} :$$

$$\mathrm{QSPP}(P, q) = \mathrm{SPP}(P, c) \quad \forall\ s\text{-}t\text{-paths } P.$$

# Example



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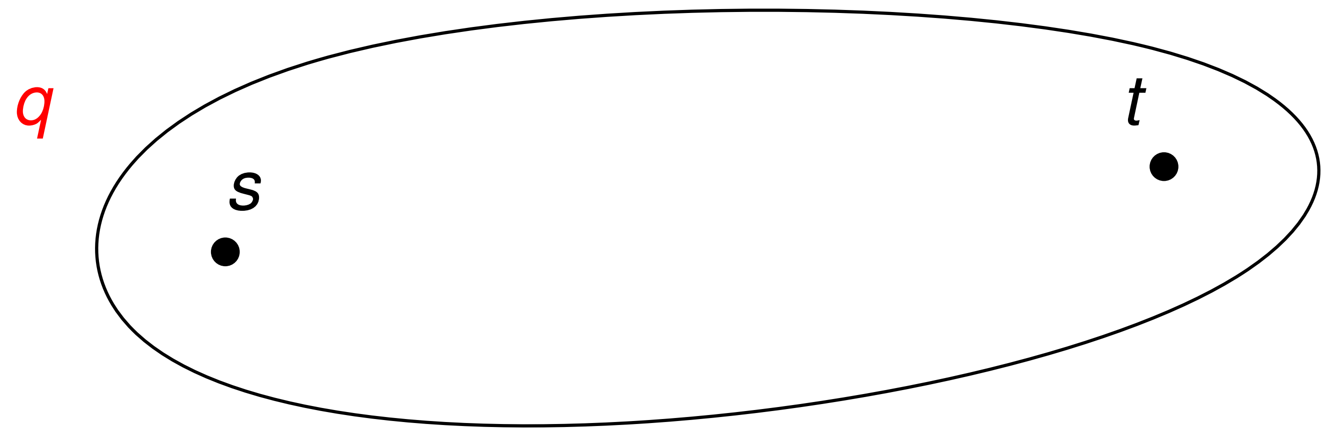


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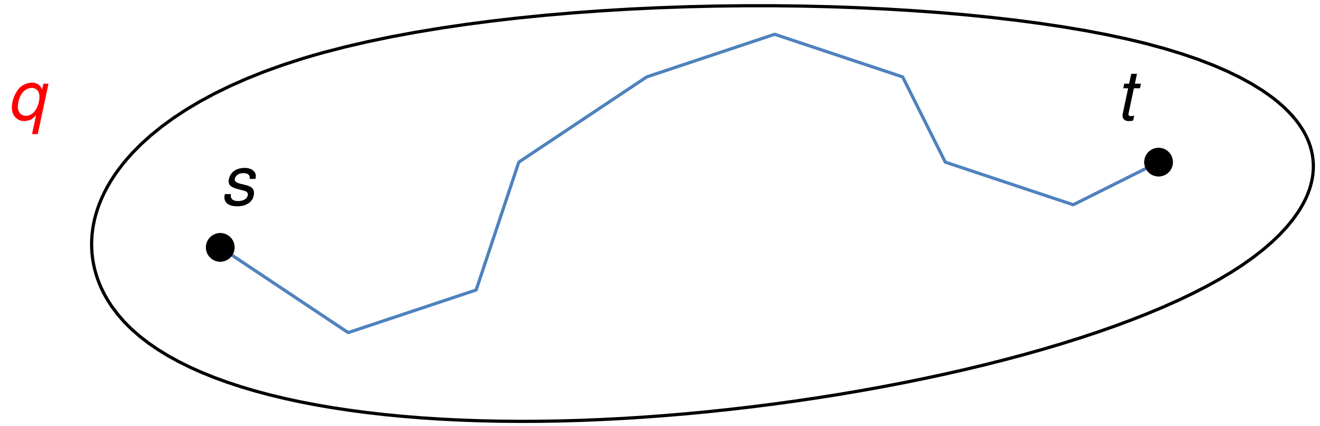


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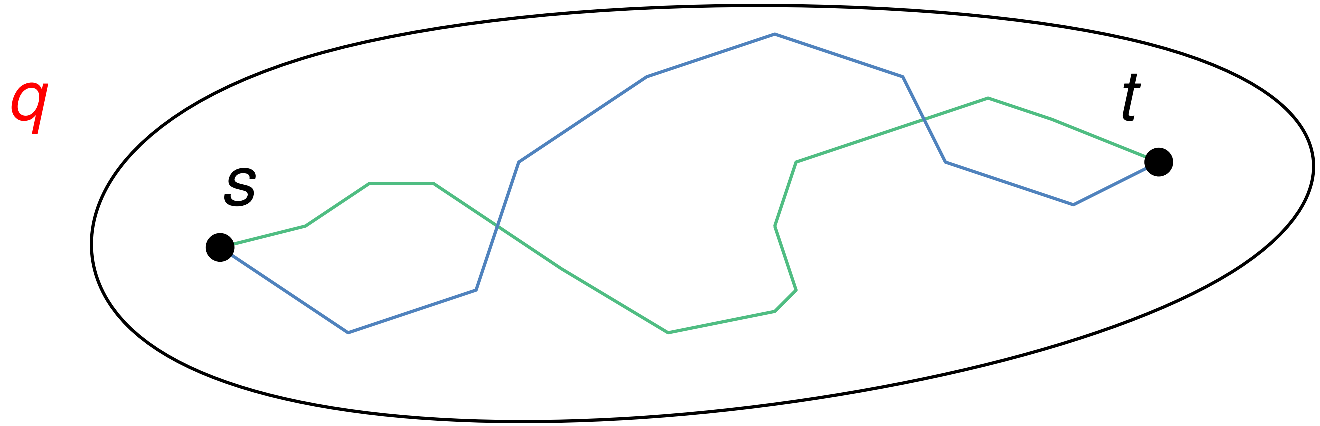


$$\exists c : E \to \mathbb{R}$$

$$\text{QSPP}(P_1, q) = \text{SPP}(P_1, c)$$

$$\text{QSPP}(P_2, q) = \text{SPP}(P_2, c)$$

$$\vdots$$

# Why do we care?

- To solve QSPP
- Appears naturally (nonnegative symm. product matrix, sum matrix)

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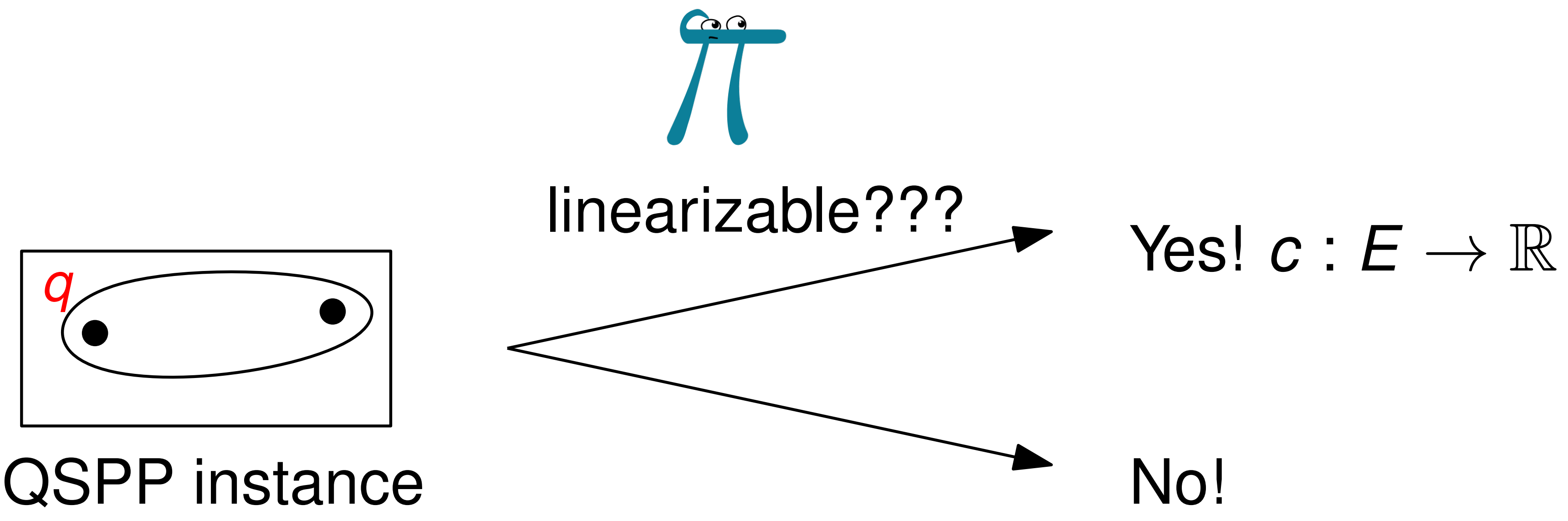
- To solve QSPP
- Appears naturally (nonnegative symm. product matrix, sum matrix)
- To understand existing and develop new branch & bound algos [Hu, Sotirov '20]

# Recognition Problem



# Previous work

- [Bookhold '90]: Introduces linearizations of quadratic problems
- Quadratic assignment [e.g. Erdogan, Tansel '06, Kabadi, Punnen '11, Cela, Deineko, Woeginger '16 ] MST [Custic, Punnen '18,Sotirov, Verchere '21] TSP [Punnen, Walter, Woods '18]

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- [Cela, Klinz, Lendl, Orlin, Woeginger, W. '21] cyclic graphs coNP-complete

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# Two research questions

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2. Generalization to order-$d$, with $d > 2$?

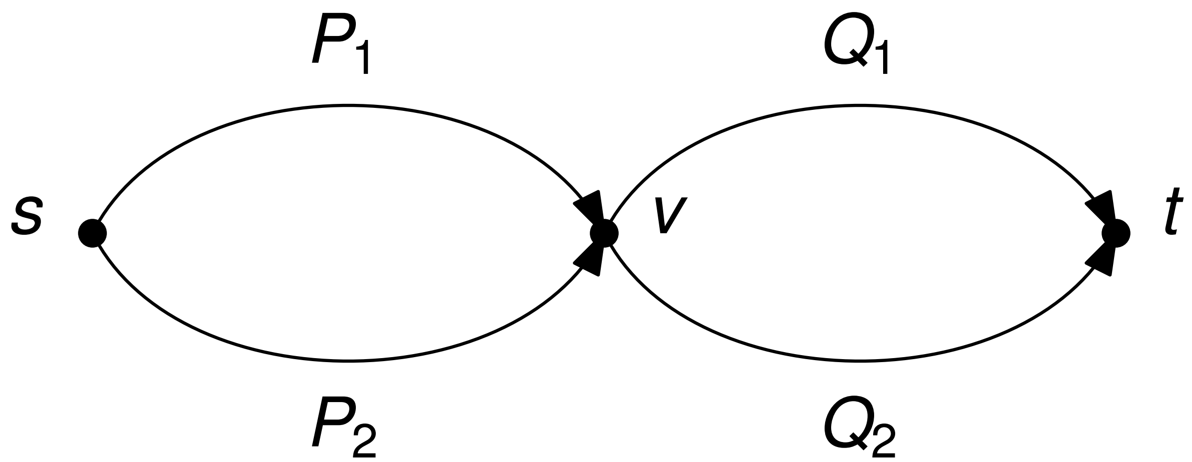
   $q : \{B \subseteq A : |B| \leq d\} \to \mathbb{R}$ **Yes!** $O(m^d)$

# Main result

**Theorem**

The lin. recognition problem of order-$d$ on acyclic digraphs can be solved in $O(m^d)$ for all $d \geq 2$.

# Two-path system



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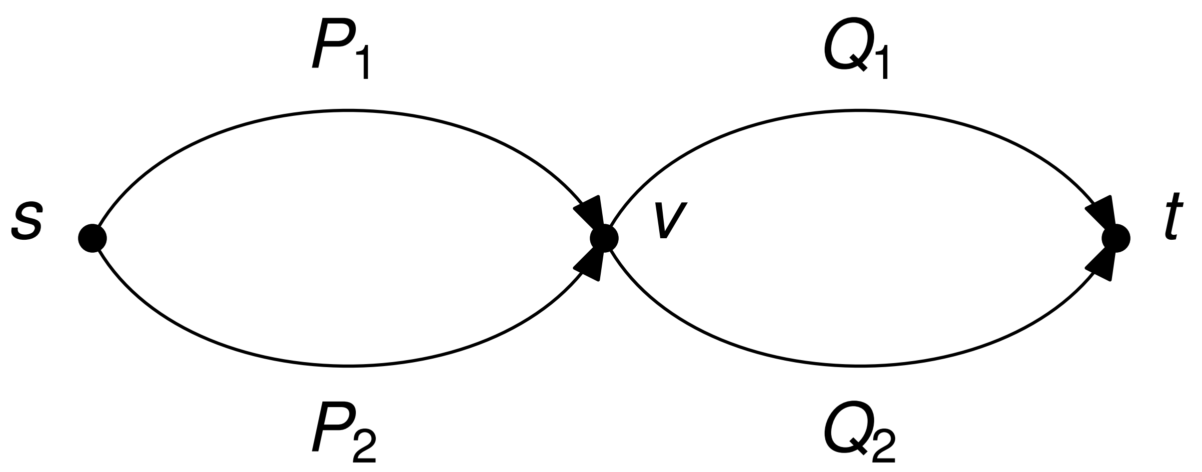


$P_1 \cdot Q_1$
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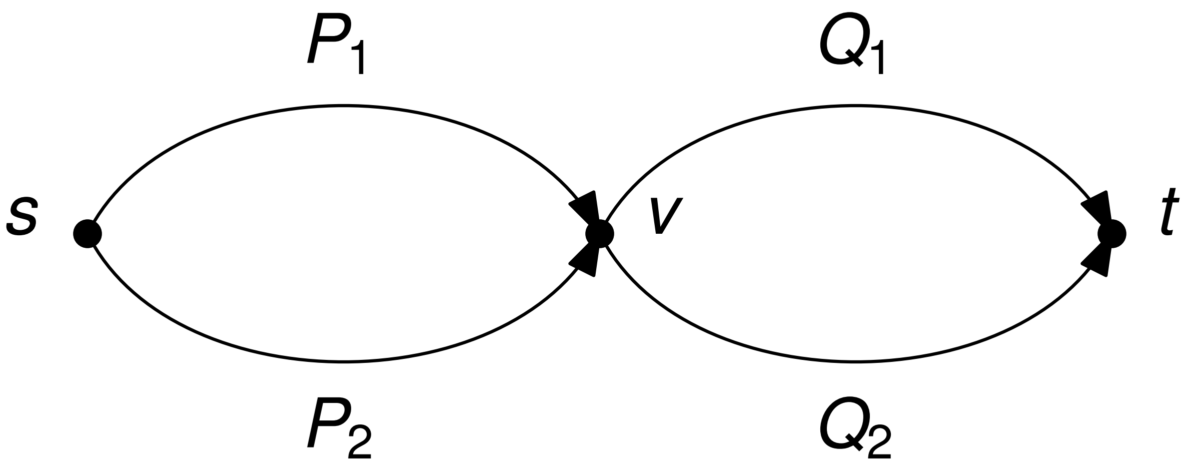


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Obs: A two-path system is linearizable iff

$$\mathrm{QSPP}(P_1 \cdot Q_1) + \mathrm{QSPP}(P_2 \cdot Q_2) = \mathrm{QSPP}(P_2 \cdot Q_1) + \mathrm{QSPP}(P_1 \cdot Q_2)$$

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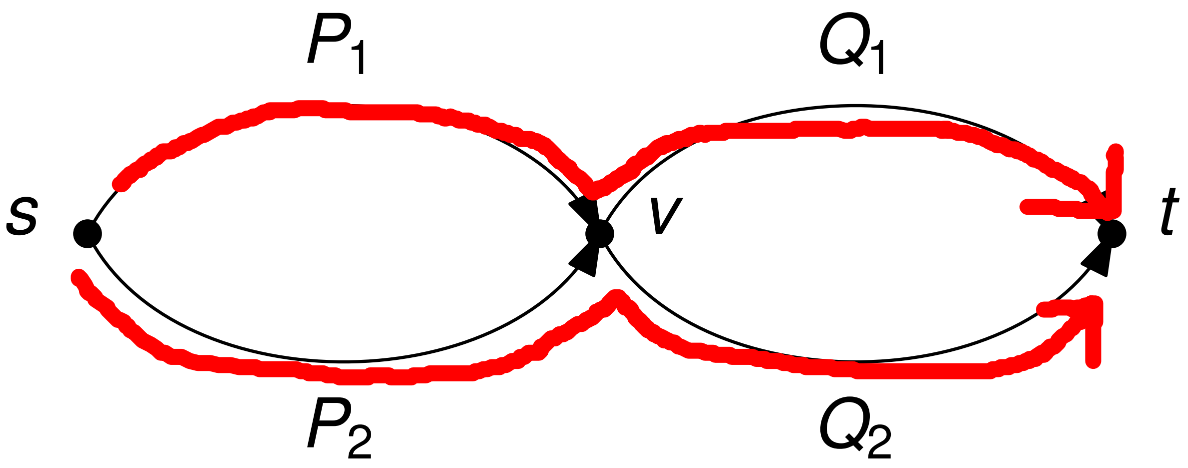


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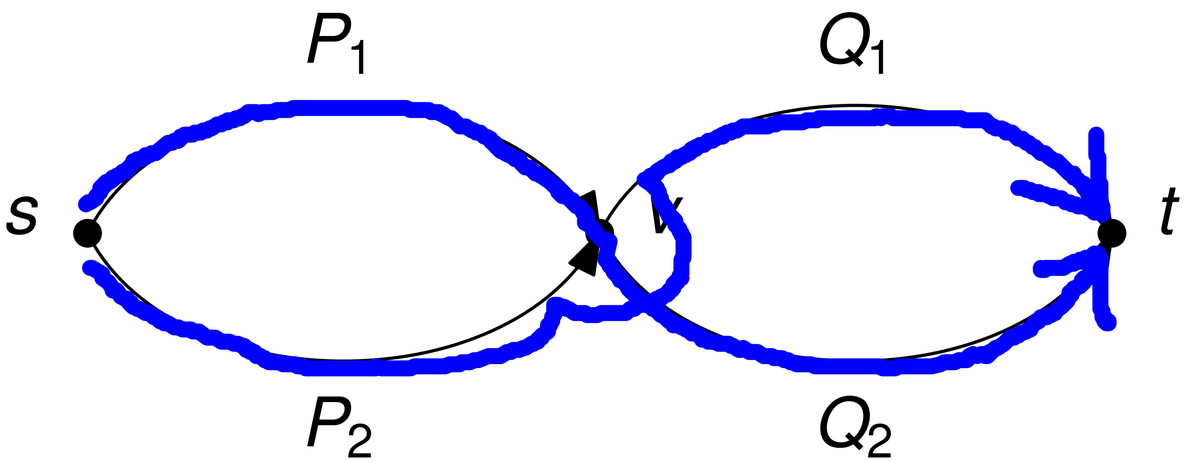


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Obvious:

$(G, q)$ linearizable $\Rightarrow$ Every two-path system in $(G, q)$ linearizable

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$(G, q)$ linearizable $\Leftarrow$ Every two-path system in $(G, q)$ linearizable ???

# Elegant characterization

**Theorem**

An acyclic QSPP instance is linearizable if and only if every two-path-system is linearizable.

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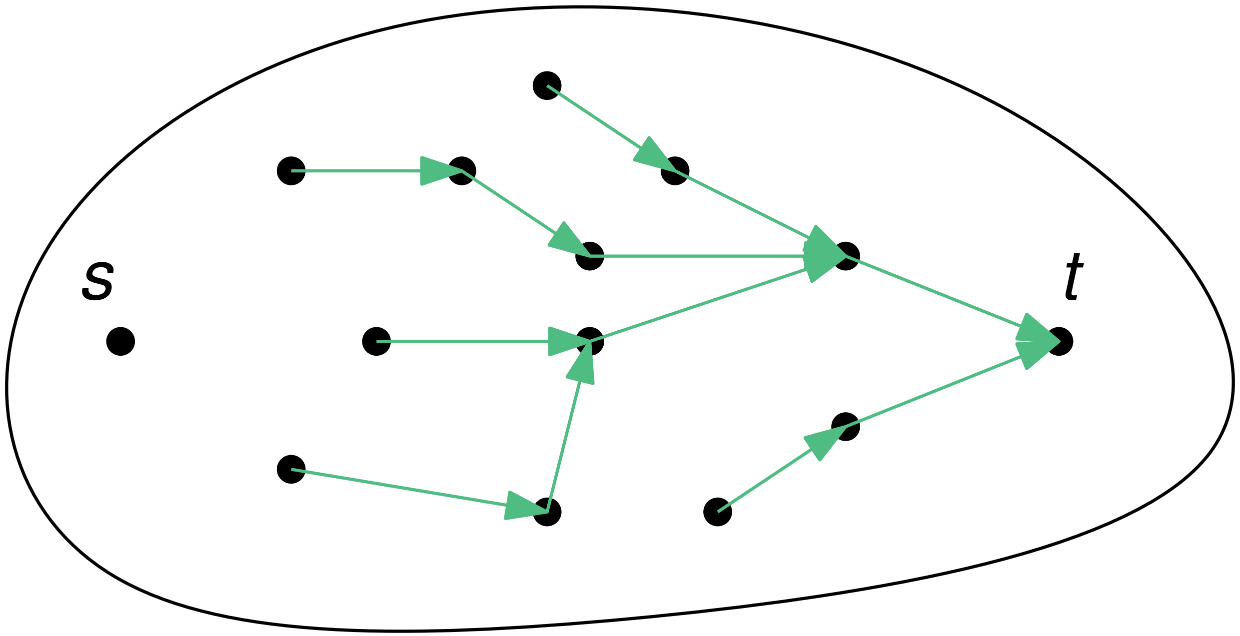
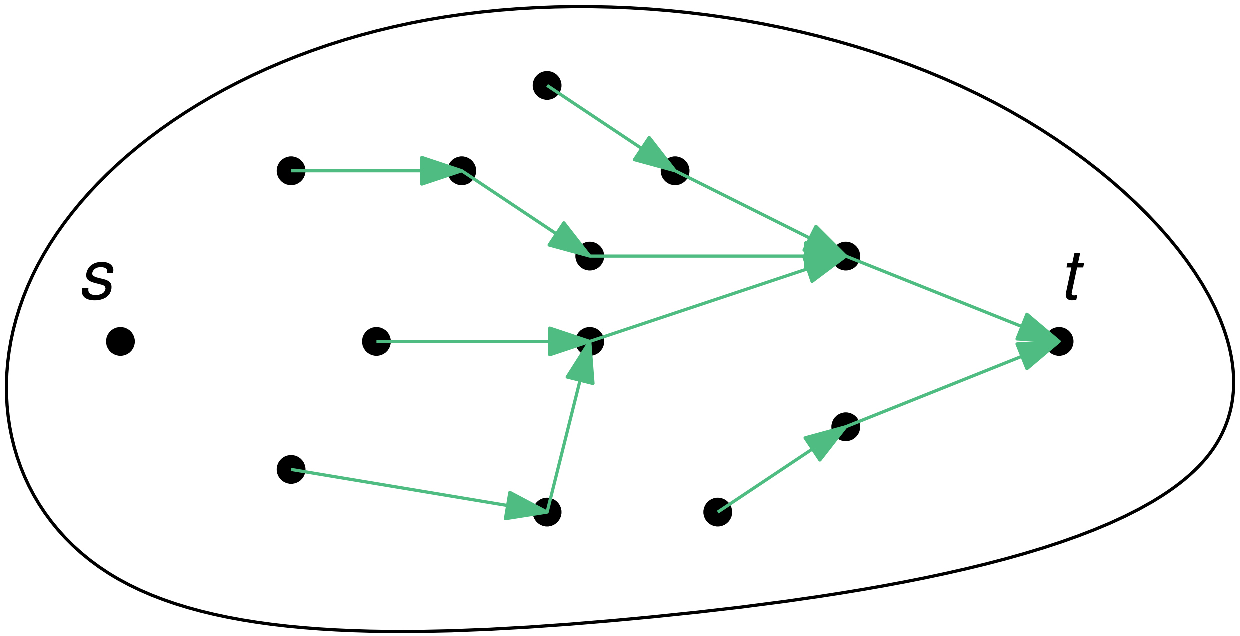
An acyclic QSPP instance is linearizable if and only if every two-path-system is linearizable.

Holds even in much more general case $f : \mathscr{P}_{s,t} \to \mathbb{R}$.

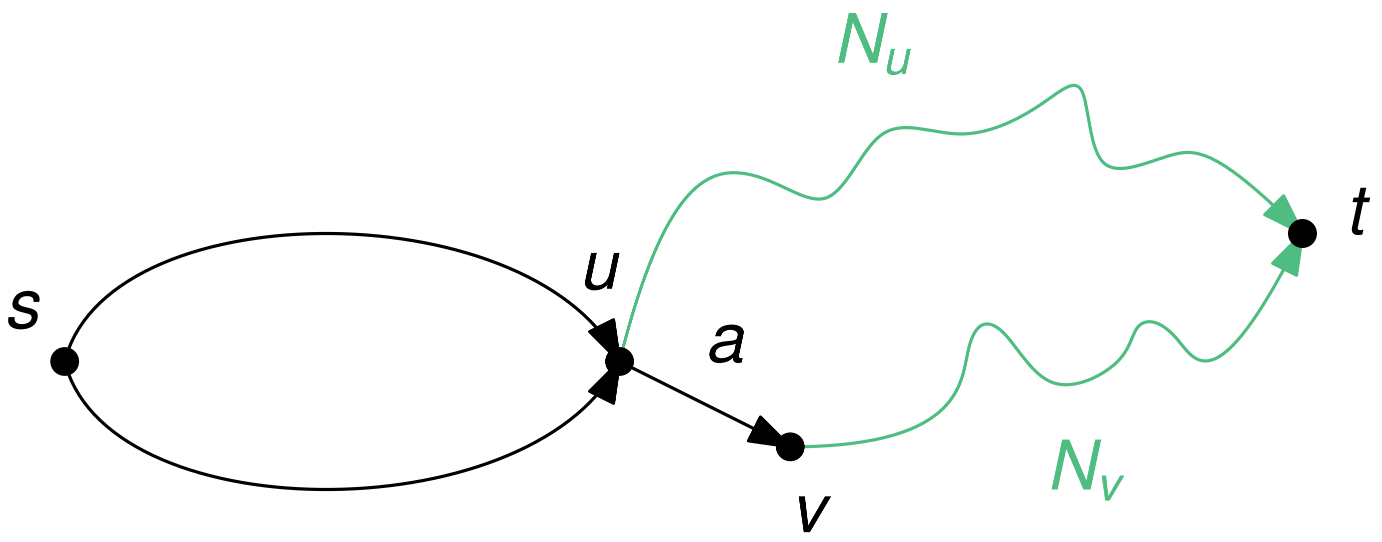



*N*: system of nonbasic arcs




$N$: system of nonbasic arcs

$$\exists_1 c : c \text{ is linearization} \wedge \; c_{|N} = 0$$


$N_u$
$t$
$u$
$s$
$a$
$v$
$N_v$

Characterization ✓

# We are done. ☺

Characterization ✓

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# No! ☹

Characterization ✓

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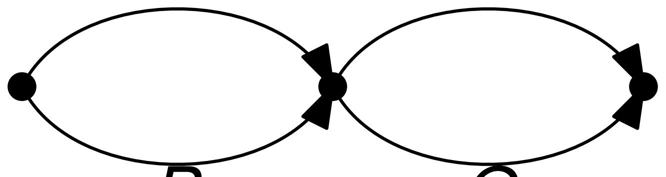
# No! ☹

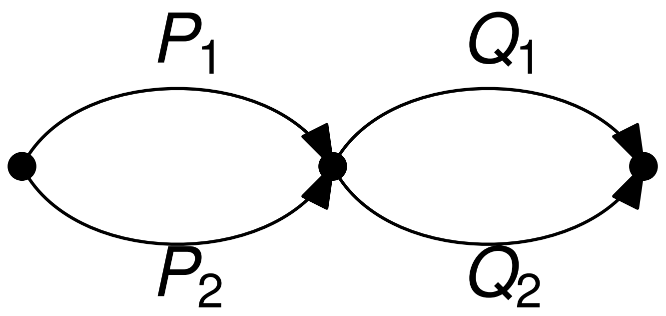
Exponentially many different two-path systems

# Solution




$$
\begin{aligned}
& \text{QSPP}(P_1 \cdot Q_1) + \text{QSPP}(P_2 \cdot Q_2) = \text{QSPP}(P_2 \cdot Q_1) + \text{QSPP}(P_1 \cdot Q_2) \\
\Leftrightarrow\ & \text{QSPP}(P_1 \cdot Q_1) - \text{QSPP}(P_1 \cdot Q_2) = \text{QSPP}(P_2 \cdot Q_1) - \text{QSPP}(P_2 \cdot Q_2) \\
\Leftrightarrow\ & \text{QSPP}(P \cdot Q_1) - \text{QSPP}(P \cdot Q_2) = \text{constant independent of } P
\end{aligned}
$$

# Solution




$$\begin{aligned}
& \mathrm{QSPP}(P_1 \cdot Q_1) + \mathrm{QSPP}(P_2 \cdot Q_2) = \mathrm{QSPP}(P_2 \cdot Q_1) + \mathrm{QSPP}(P_1 \cdot Q_2) \\
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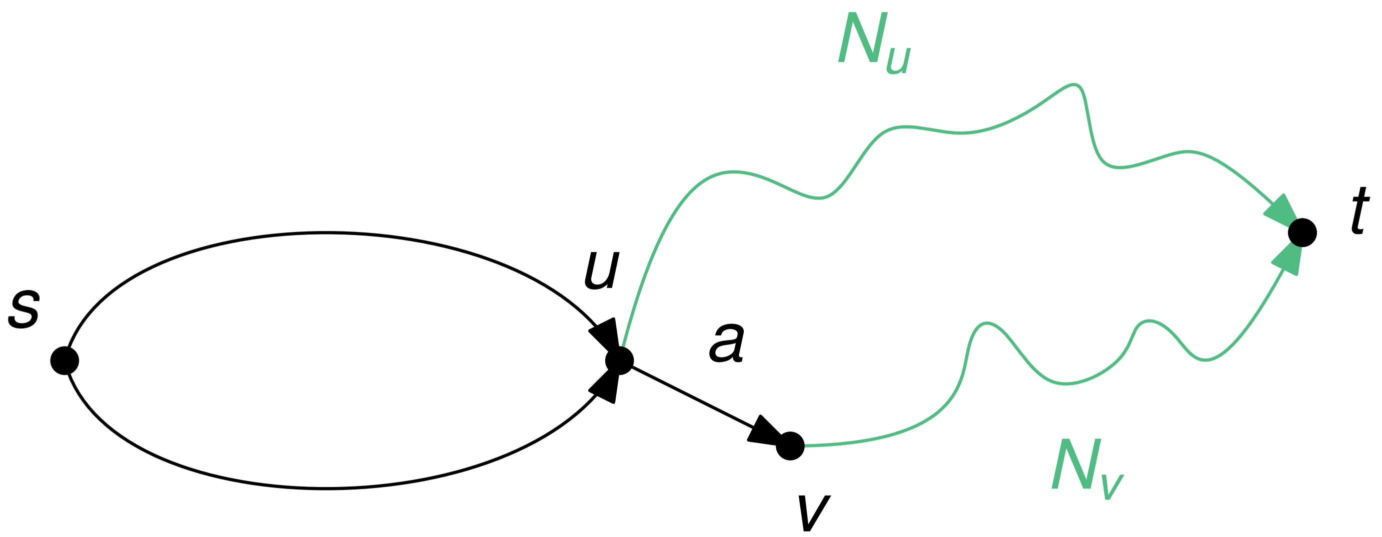
\end{aligned}$$

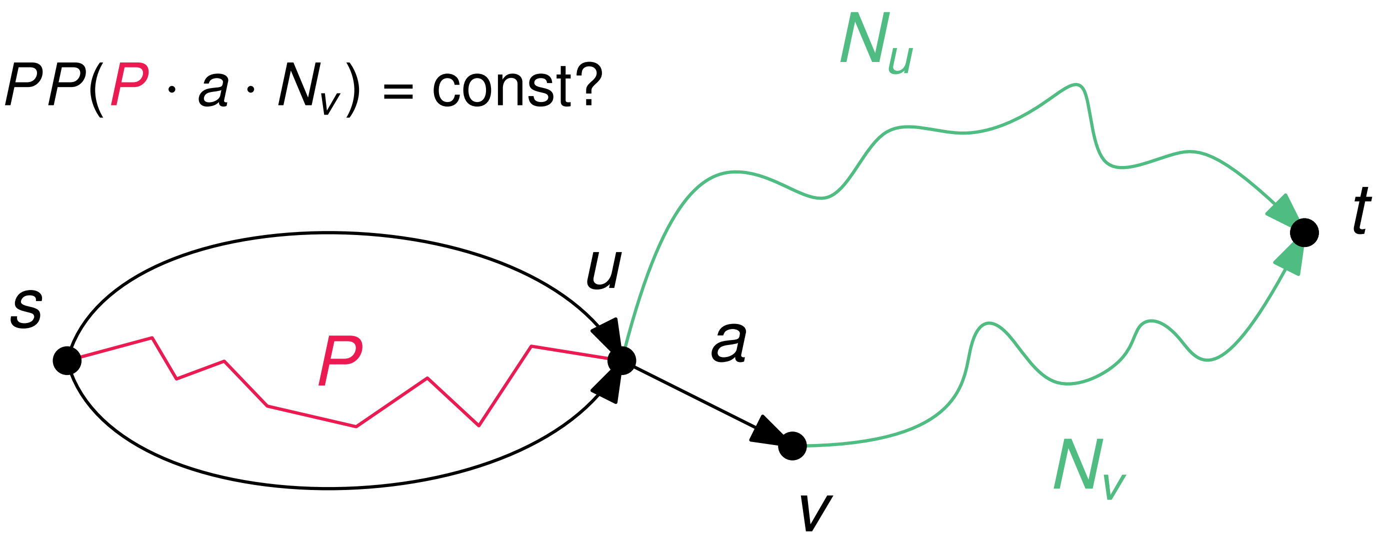
higher-order terms cancel out!

# Proof idea



# Proof idea

$\text{QSPP}(P \cdot N_u) - QSPP(P \cdot a \cdot N_v) = \text{const?}$



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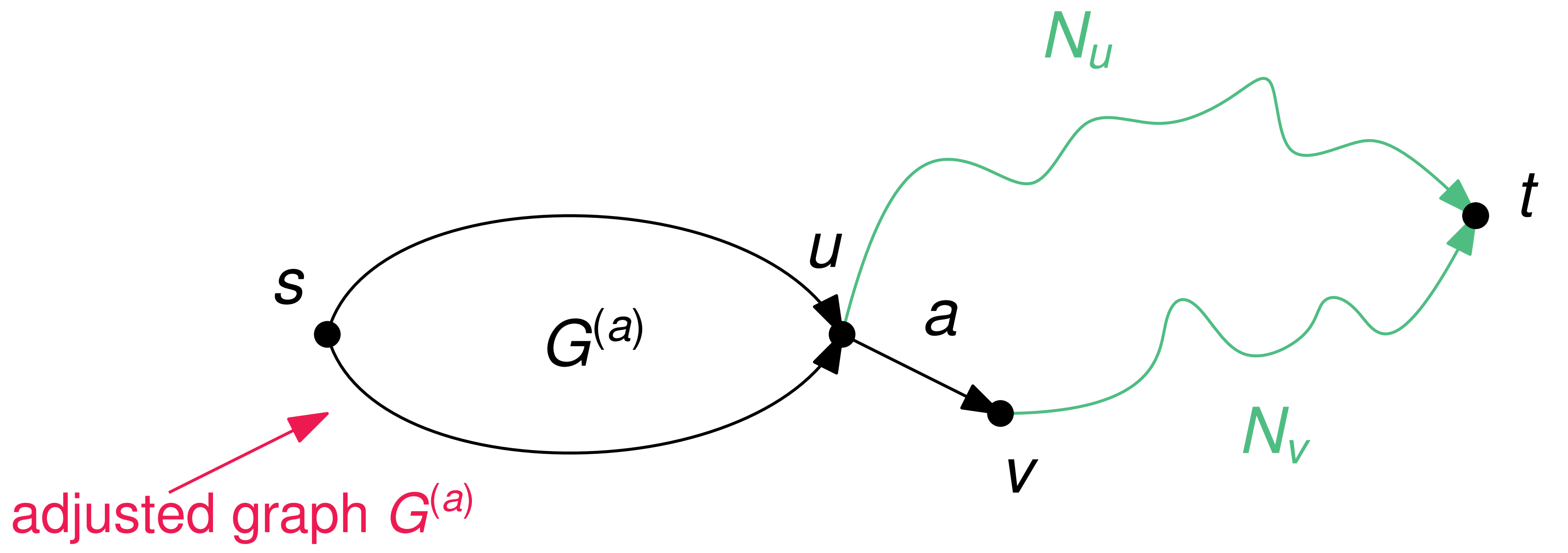

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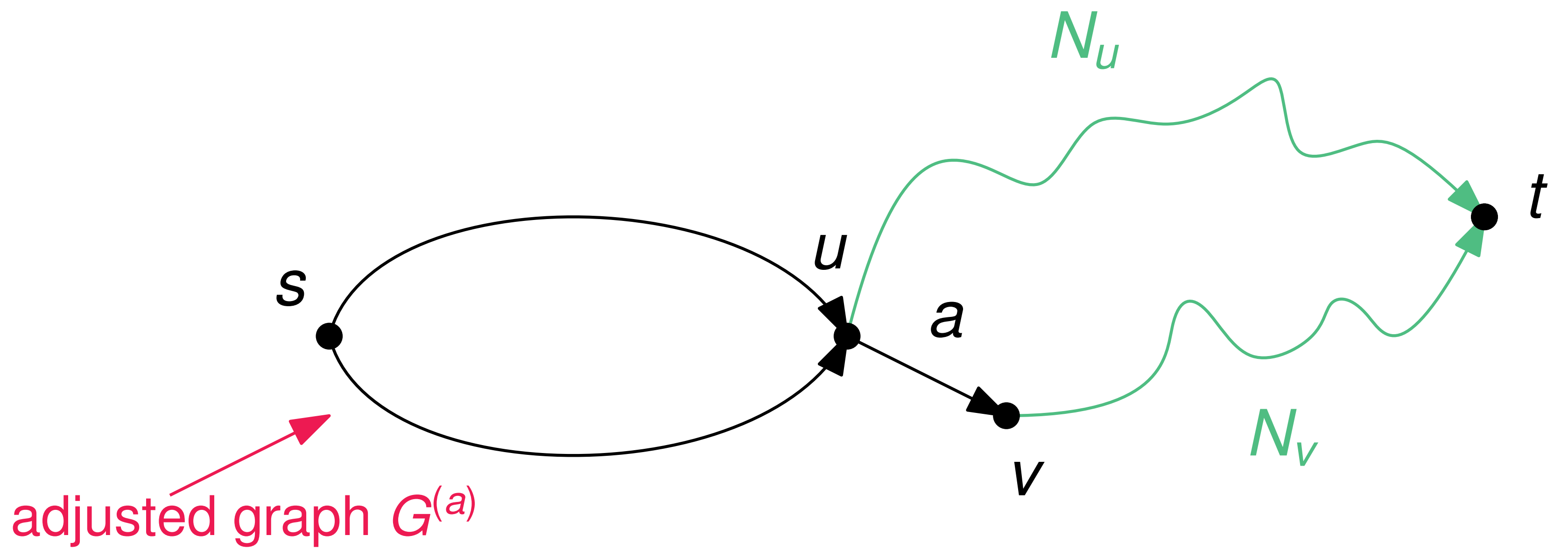


$$q^{(a)}(e) := \left( \sum_{f \in N_u} q(\{e, f\}) \right) - \left( \sum_{f \in \{a\} \cup N_v} q(\{e, f\}) \right).$$

adjusted costs $q^{(a)}$

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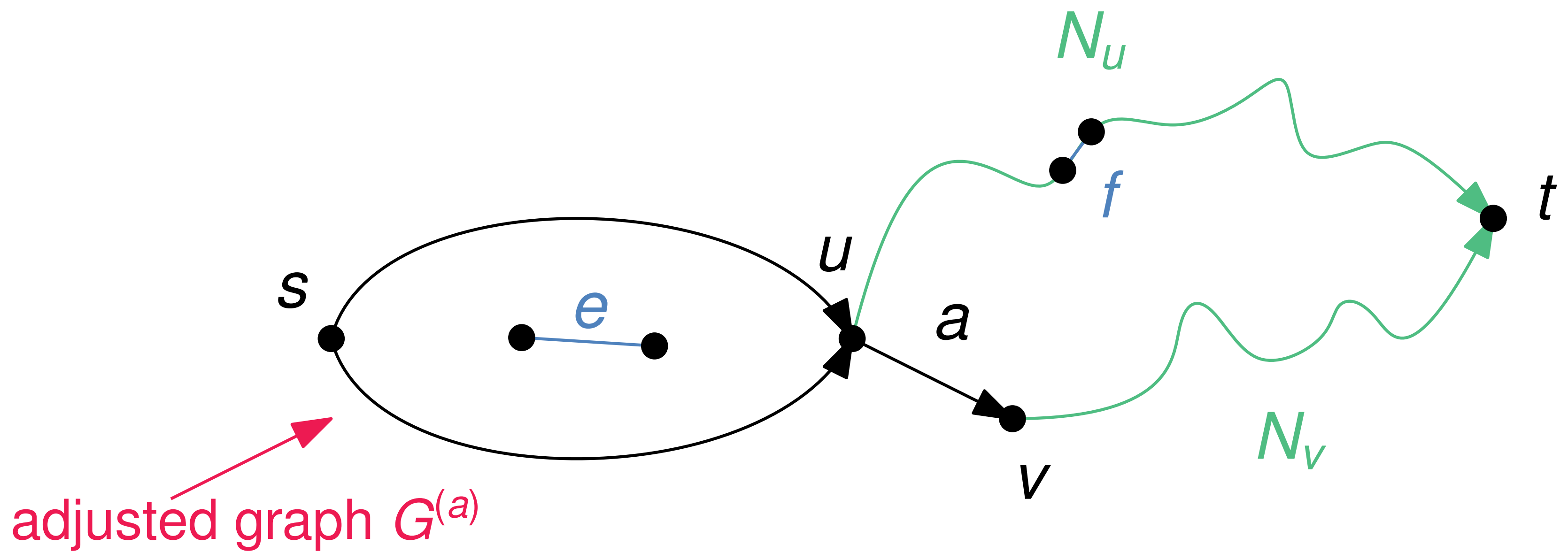


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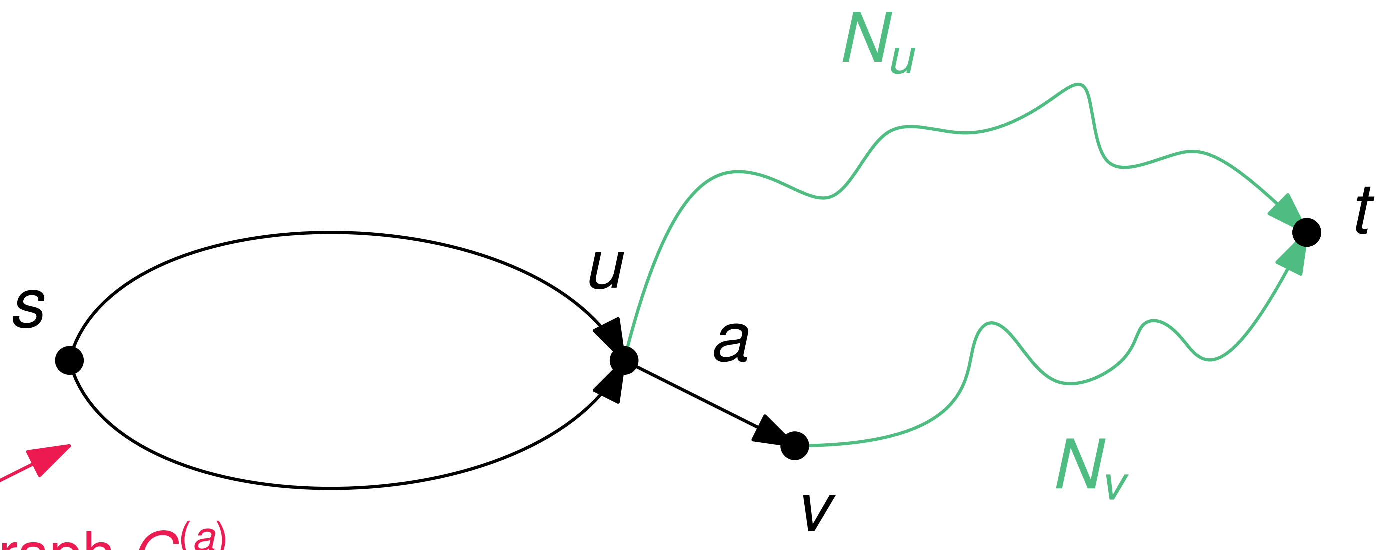


adjusted graph $G^{(a)}$

adjusted costs $q^{(a)}$

$$q^{(a)}_{d-1}(B) := \left( \sum_{\substack{C \subseteq N_u \\ |C| \leq d - |B|}} q_d(B \cup C) \right) - \left( \sum_{\substack{C \subseteq a \cdot N_v \\ |C| \leq d - |B|}} q_d(B \cup C) \right).$$

Able to show:

- $q$ quadratic $\Rightarrow q^{(a)}$ linear
- linearizable $\Leftrightarrow$ all paths in $G^{(a)}$ have identical cost $q^{(a)}(P)$.

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Exercise: How to figure out if all paths have identical (linear) cost?

Characterization ✓ Reduction ✓

We are done. ☺

Characterization ✓ Reduction ✓

We are done. ☺

No! ☹

Characterization ✓ Reduction ✓

# We are done. ☺

# No! ☹

Only $O(m^3)$ algorithm.
Evaluating formula for $q^{(a)}$ takes too long.

# A clever way to compute $q^{(a)}$

Helper values

$$\gamma(B, x) := \sum_{\substack{C \subseteq N_x \\ |C| \leq d - |B|}} q_d(B \cup C).$$

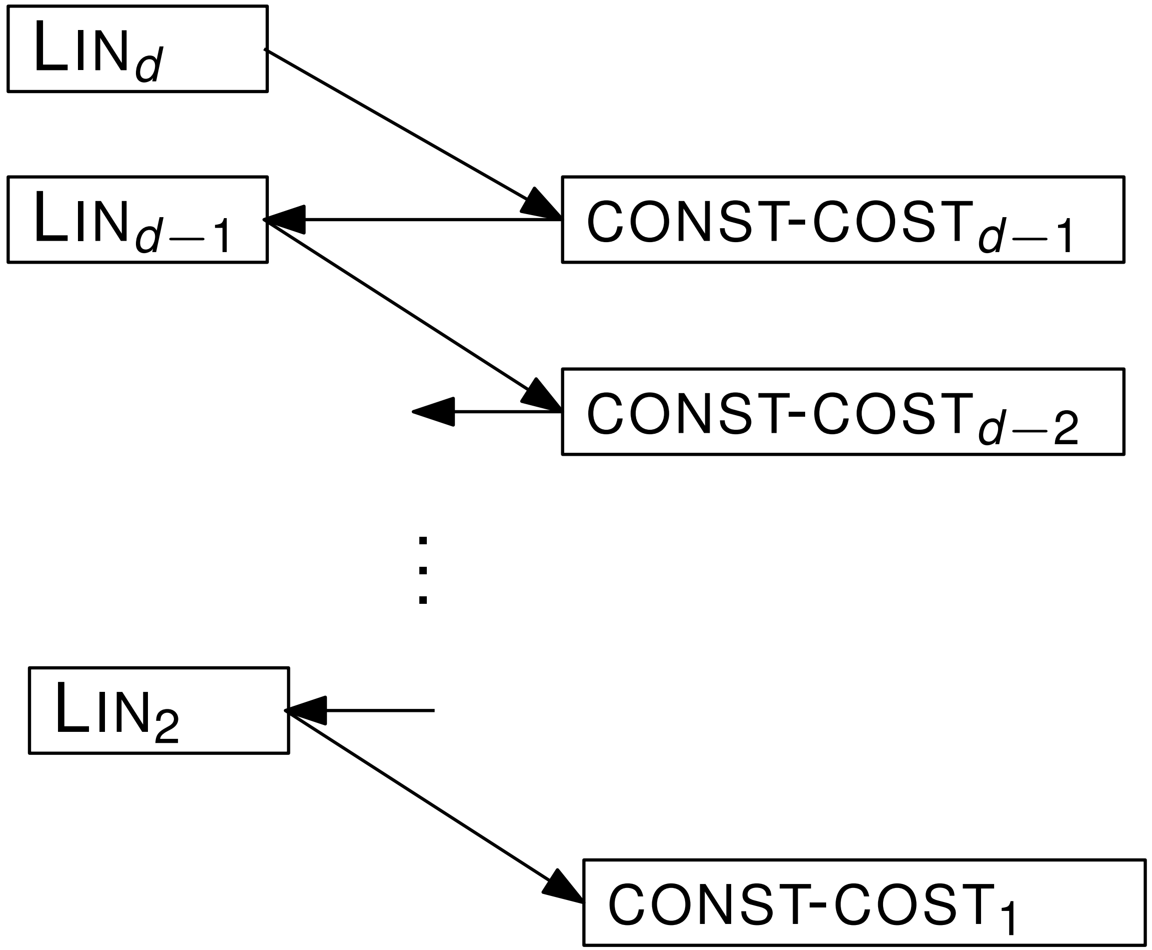
Recursive formula

$$\gamma(B, u) = \gamma(B, v) + \sum_{\substack{C \subseteq N_v \\ |C| \leq d - |B| - 1}} q_d(B \cup \{a\} \cup C).$$

Compute $q^{(a)}$ using

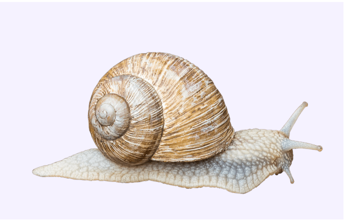
$$q^{(a)}_{d-1}(B) = \gamma(B, u) - \gamma(B, v) - \sum_{\substack{C \subseteq N_v \\ |C| \leq d - |B| - 1}} q_d(B \cup \{a\} \cup C).$$

# Everything can be generalized to $d > 2$



# Summary

- Linear time algorithm for linearizing QSPP
- Factor $O(mn)$ better than previous algo
- Based on new characterization of linearizability
- Generalizes to order $d > 2$



$\Rightarrow$

✓

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Further work

- Sparse cost matrices
- Computational experiments

$\Rightarrow$

$\checkmark$