An Introduction to Semidefinite Program Relaxations of Quadratically Constrained Quadratic Programs

(Problem Session)

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Question 1. Recall that $\mathbb{S}^n_+$ is the cone of $n \times n$ positive semidefinite symmetric matrices, i.e.,

$$\mathbb{S}^n_+ = \{X \in \mathbb{S}^n : \ u^\top X u \geq 0 \ \ \forall u \in \mathbb{R}^n\}.$$

For any $X \in \mathbb{S}^n_+$, show that there exists $Y \in \mathbb{S}^n_+$ such that $X = Y^2$ (here Y is is called the *square root* of X).

> *Hint:* Can you prove this when X is diagonal? For the general case, consider using the spectral decomposition $UDU^\top$ of X, and remember that an $n \times n$ matrix U is orthogonal if and only if $U^\top U = UU^\top = I_n$.

Question 2. Show that

- $\mathbb{S}^n_+$ is a cone, and
- $(\mathbb{S}^n_+)_* = \mathbb{S}^n_+$, i.e., $\mathbb{S}^n_+$ is self-dual. (Recall that given a cone $\mathbb{K}$, we define its dual cone as $K_* := \{y : \ \langle x, y\rangle \geq 0, \forall x \in \mathbb{K}\}$.)

  Hint: For the second part, recall that $X \in \mathbb{S}^n_+$ if and only if $X = \sum_{i\in[k]} x_k x_k^\top$ for some vectors $x_k \in \mathbb{R}^n$.

Question 3. Consider the space of $n \times n$ symmetric matrices, i.e., $\mathbb{E} = \mathbb{S}^n$ equipped with the Frobenius inner product $\langle X, Y\rangle = \mathrm{tr}(XY) = \sum_{i=1}^n \sum_{j=1}^n X_{ij} Y_{ij}$. Prove that $X, Y \in \mathbb{S}^n_+$ are *complementary*, i.e., $\langle X, Y\rangle = 0$, if and only if their matrix product is zero, i.e., $XY = YX = 0$. In particular, matrices from a complementary pair commute and therefore share a common orthonormal eigenbasis.

Hint: For the easy direction, use the definition of trace, $\mathrm{tr}(M)$, as the sum of diagonal entries of the matrix M. For the reverse direction, recall that $X \in \mathbb{S}^n_+$ if and only if $X = \sum_{i\in[k]} x_k x_k^\top$ for some vectors $x_k \in \mathbb{R}^n$.

Question 4. Consider the following SDP:

$$\mathrm{Opt}(P) = \inf_{X\in\mathbb{S}^3} \left\{ X_{11} : \begin{array}{c} X_{22} = 0, \\ X_{11} + 2X_{23} = 1, \\ X \succeq 0 \end{array} \right\}$$

- What is the problem data for this problem? Write it in our standard primal SDP form by defining C, A_i, b_i.
- Is this problem strictly feasible?
- What is the optimum objective value?
- Is the optimum objective value achieved?
- What is the dual problem?
- Is the dual strictly feasible? What is the dual optimum objective value? Is there a dual solution achieving the optimum objective value?

Question 5. Prove *Schur Complement Lemma*: Consider an $n \times n$ symmetric matrix

$$M := \begin{pmatrix} P & Q^\top \\ Q & R \end{pmatrix}$$

such that $R \in \mathbb{S}^k_{++}$. Then, $M \in \mathbb{S}^n_+$ (or $\mathbb{S}^n_{++}$) if and only if the matrix $P - Q^\top R^{-1} Q \in \mathbb{S}^{n-k}_+$ (or $\mathbb{S}^{n-k}_{++}$).

Hint: Recall a symmetric matrix M is positive semidefinite if and only if $a^\top M a \geq 0 \ \ \forall a \in \mathbb{R}^n$. Starting from this, set up the characterization of $M \in \mathbb{S}^n_+$ as a quadratic optimization problem and partition the decision variables into groups based on the partition of the matrix M to obtain a partial minimization form. Can you characterize the optimum solution of the partial minimization problem?

Question 6. (a) Prove the following matrix version of the scalar inequality $ab \le \frac{a^2+b^2}{2}$. Given two $m \times n$ matrices A, B, prove that the following inequality holds:

$$AB^\top + BA^\top \preceq AA^\top + BB^\top.$$

Hint: The simplest way to prove the scalar inequality $ab \le \frac{a^2+b^2}{2}$ is to observe that it is equivalent to $0 \le \frac{(a-b)^2}{2}$. Can you think of something analogous for the matrix inequality?

(b) Let I_k denote the $k \times k$ identity matrix, and let A be an $m \times n$ matrix. Prove that the following three properties are equivalent to each other:

(i) $A^\top A \preceq I_n$;

(ii) $AA^\top \preceq I_m$;

(iii) $\begin{bmatrix} I_m & A \\ A^\top & I_n \end{bmatrix} \succeq 0.$

Question 7. Let $\mathcal{S} \subseteq \mathbb{S}^n_+$ be a closed convex cone. Prove the following statements:

(a) For $X \neq 0$, $\mathbb{R}_+ X$ (the ray generated by the matrix X) is an extreme ray of $\mathcal{S}$ if and only if for every Y,

$$[X - Y, X + Y] \subseteq \mathcal{S} \implies \exists\, \alpha \in \mathbb{R} \text{ such that } Y = \alpha X.$$

(b) If $X \in \mathcal{S}$ has $\operatorname{rank}(X) = 1$, then $\mathbb{R}_+ X$ is an extreme ray of $\mathcal{S}$.

(c) $\mathcal{S}$ is ROG if and only if for each extreme ray $\mathbb{R}_+ X$ of $\mathcal{S}$ we have $\operatorname{rank}(X) = 1$. (Recall that $\mathcal{S}$ is called *rank-one generated (ROG)* if $\mathcal{S} = \operatorname{conv}\left(\mathcal{S} \cap \left\{zz^\top : z \in \mathbb{R}^{n+1}\right\}\right)$.)

Question 8. Given $\mathcal{M} \subseteq \mathbb{S}^{n+1}$, recall $\mathcal{S}(\mathcal{M}) := \left\{Z \in \mathbb{S}^{n+1}_+ : \langle M, Z\rangle \le 0, \forall M \in \mathcal{M}\right\}$. Prove the following statement which implies S-lemma:

The set $\mathcal{S}(\{M\})$ for any $M \in \mathbb{S}^{n+1}$ is ROG.

Hint: Note that here $|\mathcal{M}| = 1$. In this case, we have $\mathcal{S}(\mathcal{M})$ is ROG if and only if $\mathcal{T}(\mathcal{M}) := \left\{Z \in \mathbb{S}^{n+1}_+ : \langle M, Z\rangle = 0, \forall M \in \mathcal{M}\right\}$ is ROG. Show that $\mathcal{T}(\mathcal{M})$ is ROG by contradiction. In particular, assume there is an extreme with rank at least 2, and examine its eigenvalue decomposition while taking into account that $x^\top M x$ is a continuous function of x.

Question 9. Given $A \in \mathbb{S}^n$, consider the set $Q_A = \{x \in \mathbb{R}^n : \ x^\top Ax \leq 0\}$.

(a) Let $B \in \mathbb{S}^n$ be such that $B \neq A$ and $Q_B = Q_A$. Then, is it always true that there exists $\rho > 0$ such that $B = \rho A$?

(b) Suppose that $A \in \mathbb{S}^n$ satisfies $A_{ij} \geq 0$ for all i, j. Under this condition, does your answer to part **(a)** change?

(c) Suppose that $A \in \mathbb{S}^n$ satisfies $\lambda_{\min}(A) < 0 < \lambda_{\max}(A)$. Under this condition, does your answer to part **(a)** change?

Hint: This exercise is designed to point out whether we can guarantee some sort of uniqueness of representation in the case of quadratic functions. The first two statements are incorrect. For the last one, think about how you can use S-lemma.