Monoidal Strengthening and Unique Lifting in MIQCPs

Antonia Chmiela, Gonzalo Muñoz, Felipe Serrano

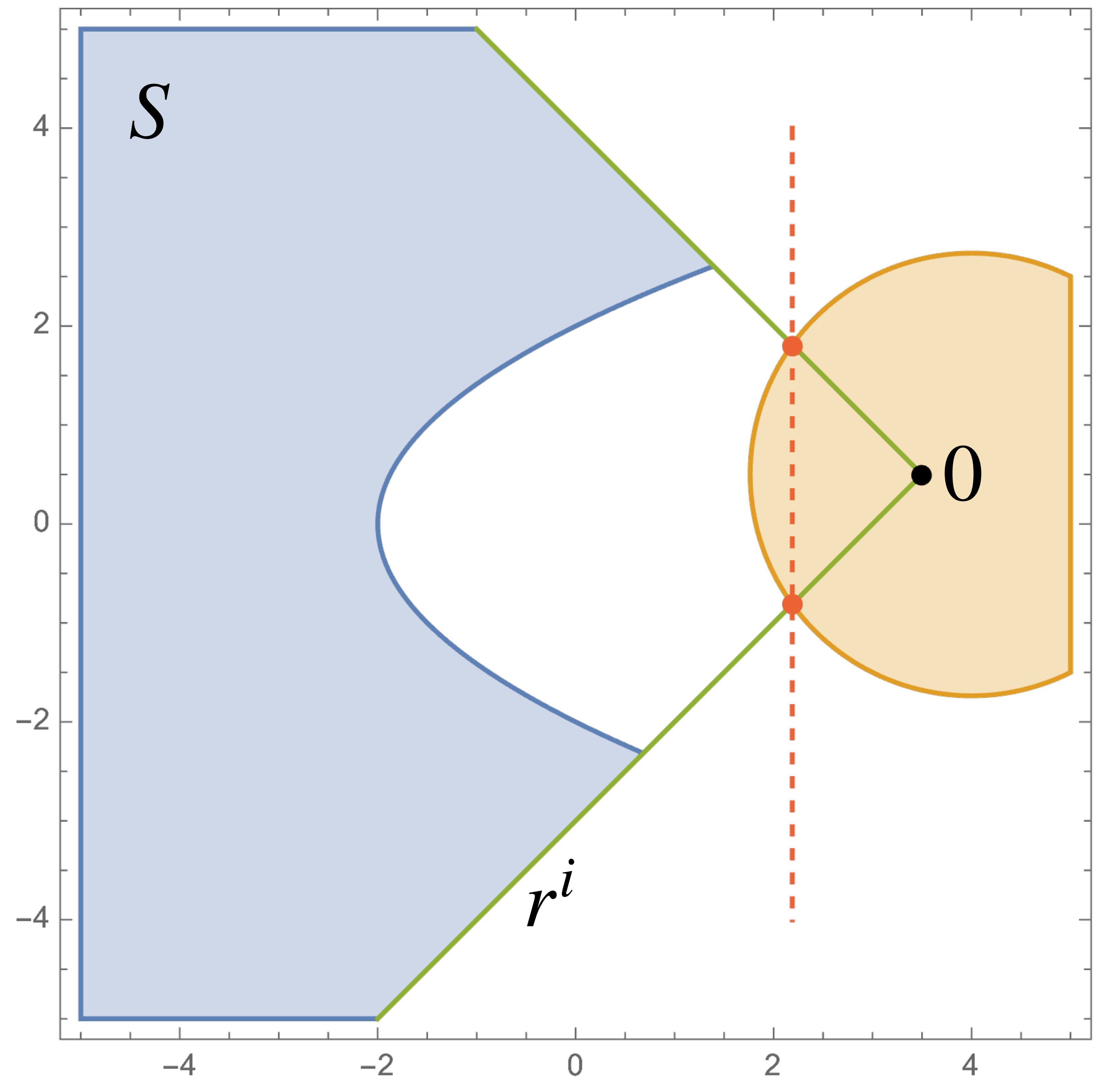
IPCO 2023 — June 23, 2023

Intersection Cuts

Formalization

- S closed set, $0 \notin S$ and

$$\sum r^i x_i \in S,\ \ x_i \geq 0$$



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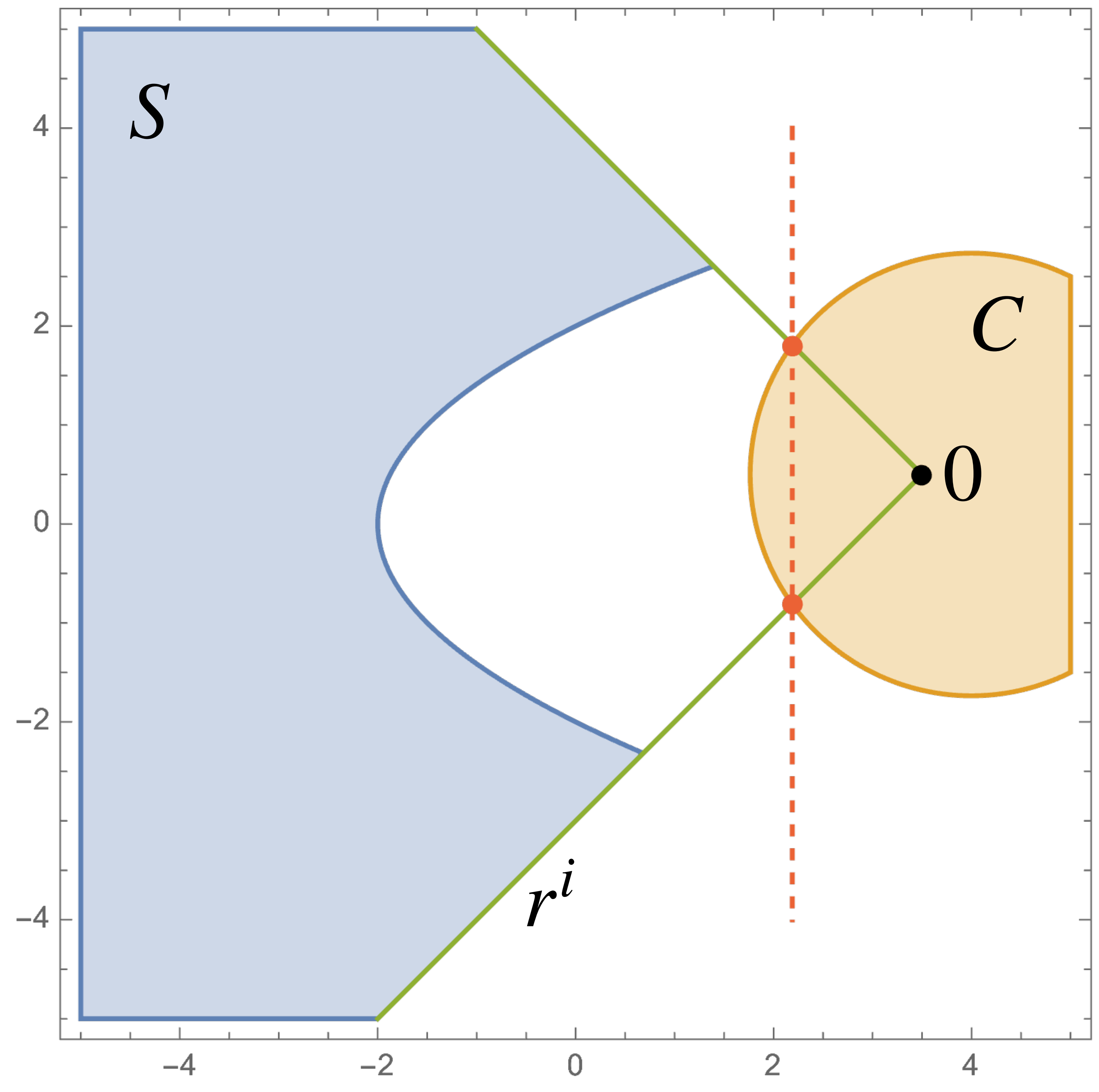
$$C = \{x \mid \phi(x) \leq 1\}$$

with ϕ sublinear



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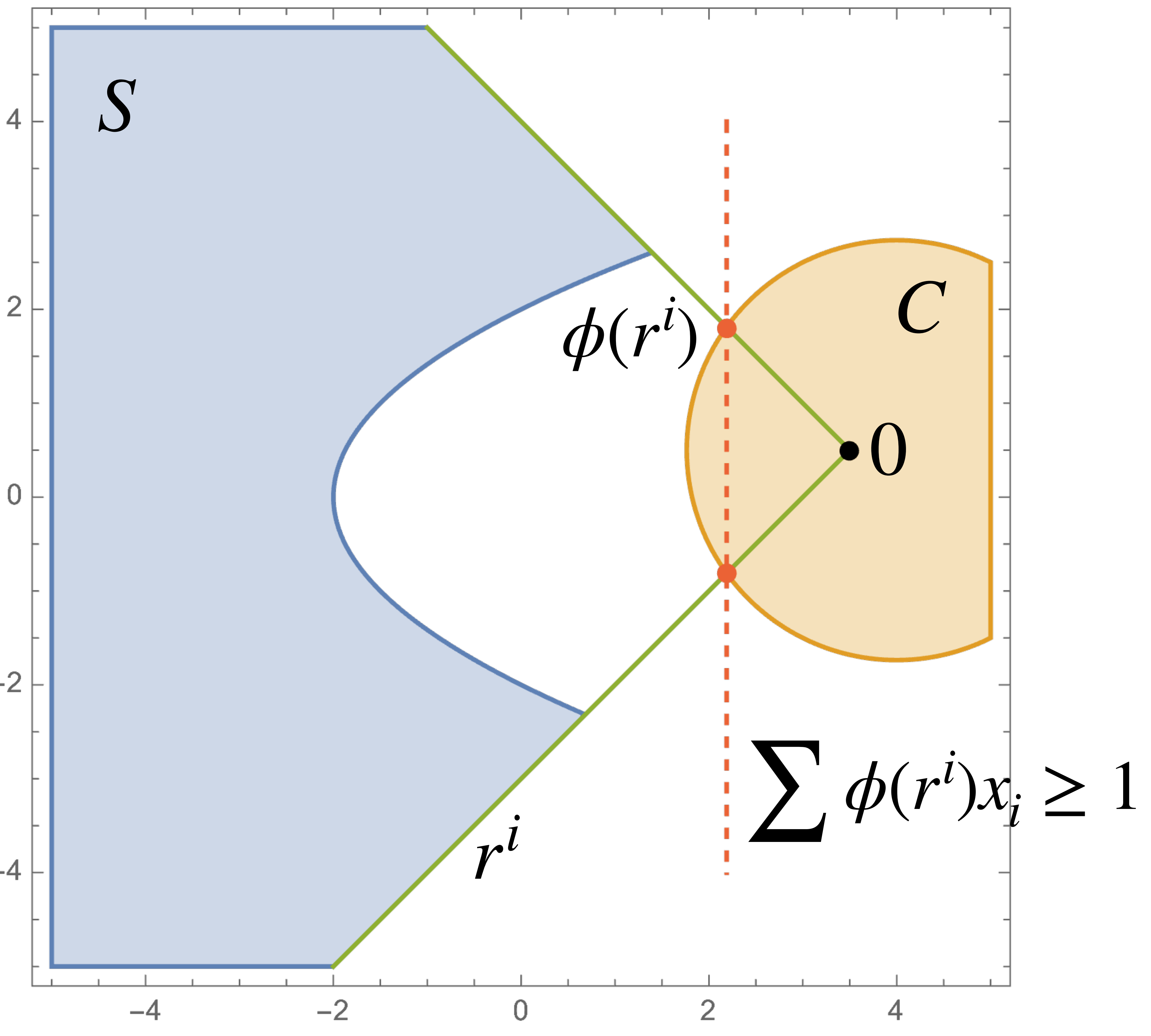

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with ϕ sublinear

$$\phi(x) \geq 1, \forall x \in S$$
$$\Rightarrow\ \ \phi(\sum r^i x_i) \geq 1, \forall x \in S$$

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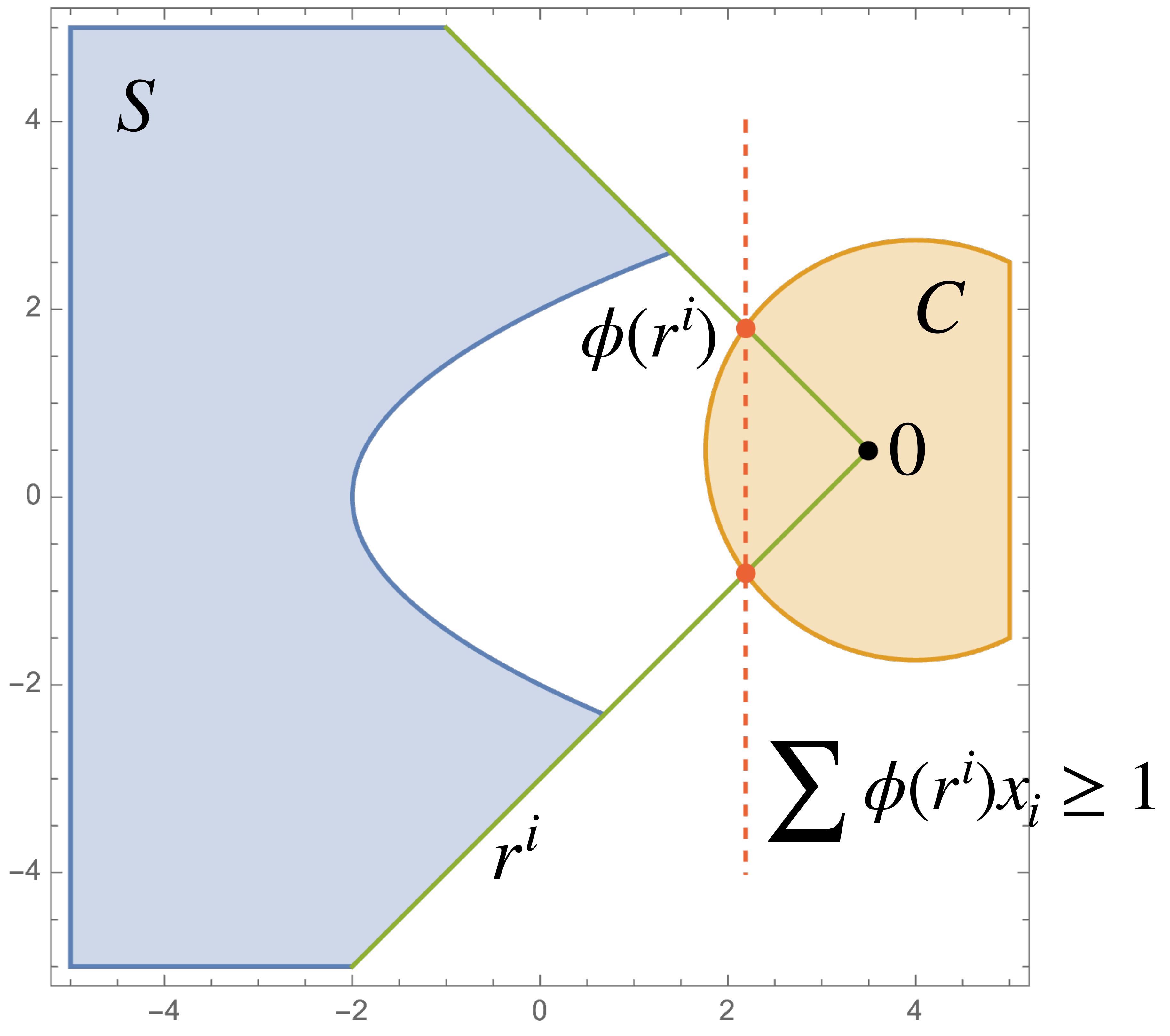

Improving the Cut by Exploiting Integrality

Monoidal strengthening*

Assume $x_i \in \mathbb{Z}_+$. Let M be a monoid such that C is $(S + M)$-free

*E. Balas, R. Jeroslow, Strengthening cuts for mixed integer programs, 1980

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Monoidal strengthening

closed under addition and contains a neutral element

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$$\sum \inf_{m \in M} \phi(r^i + m) x_i \geq 1$$

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add $\sum m^i x_i$ to both sides

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element of M

integer and non-negative

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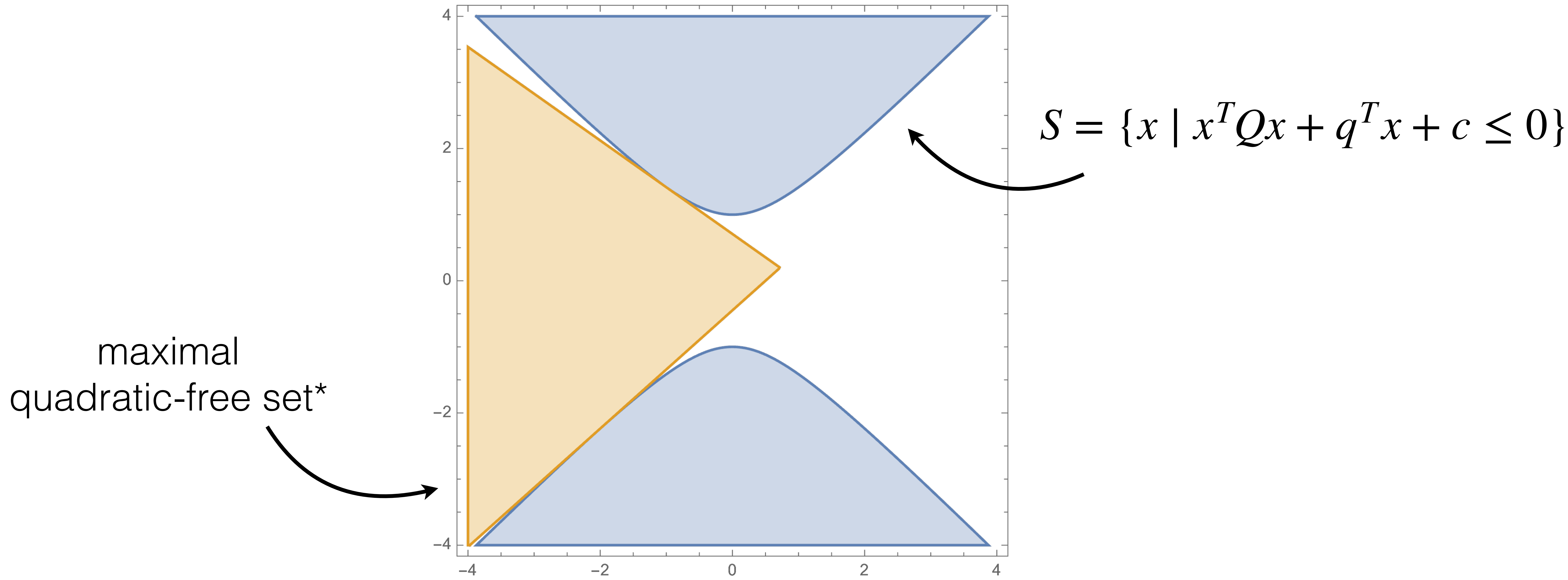
$M + M = M$

Intersection Cuts for MIQCPs

Maximal quadratic-free sets*



*G. Muñoz and F. Serrano, Maximal quadratic-free sets, 2021

Applying Monoidal Strengthening

Next steps

1. Find a monoid M such that C is $(S + M)$-free

2. Find the best cut coefficient by solving

$$\inf_{m \in M} \quad \phi(r + m)$$

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Looking at monoidal strengthening from another perspective

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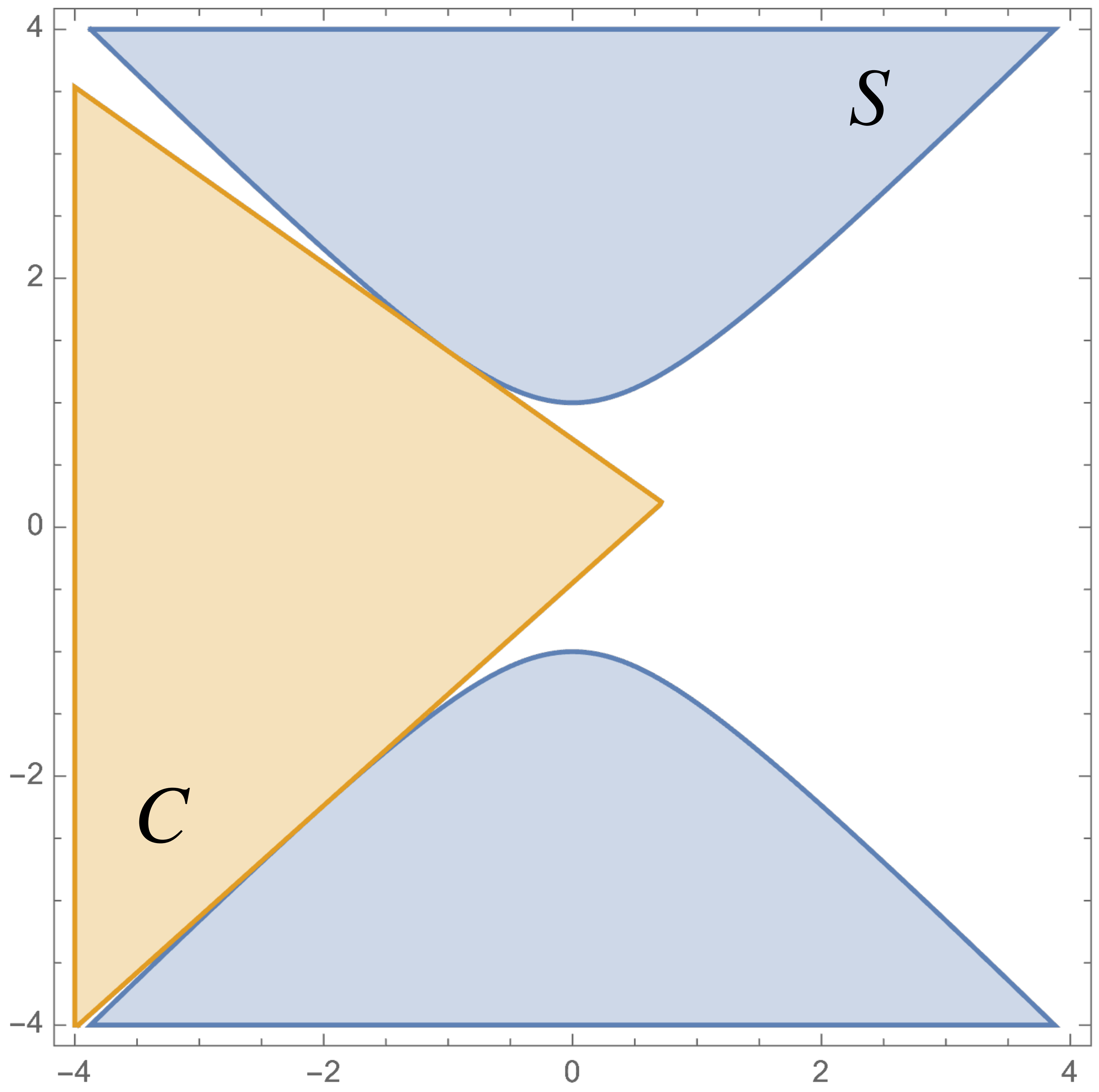

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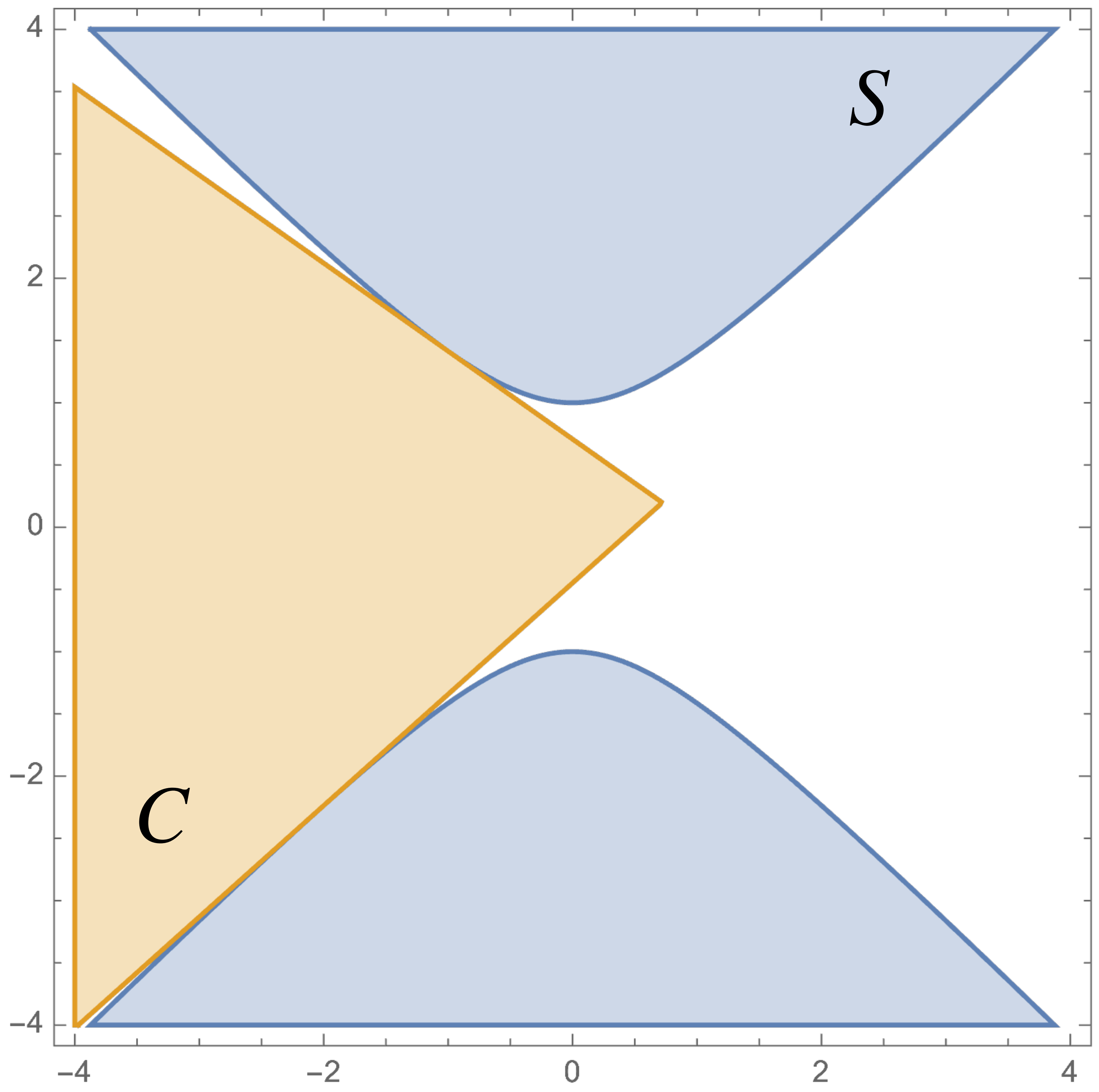

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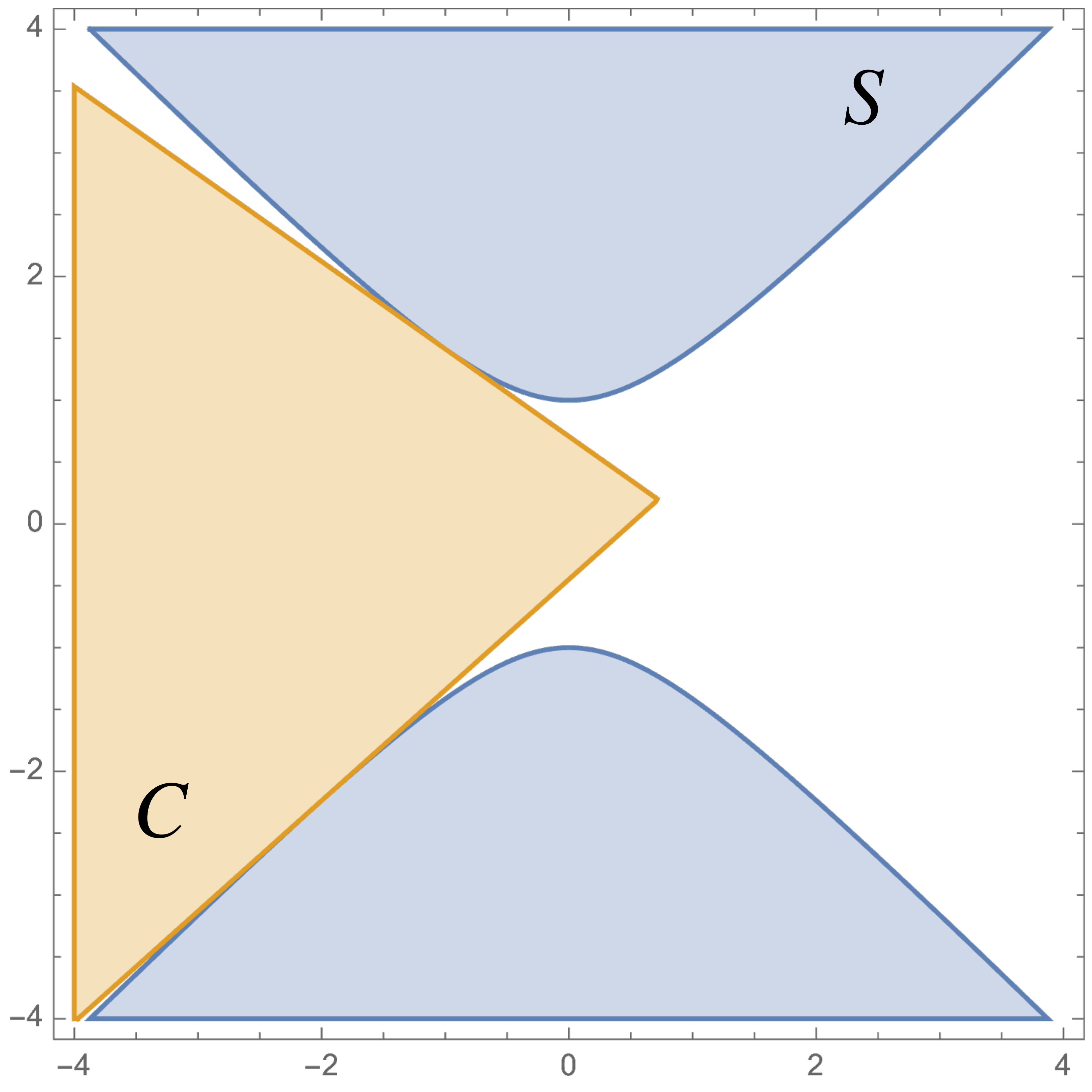
$\Rightarrow$ $-M$ gives us directions by which we can translate C without intersecting S



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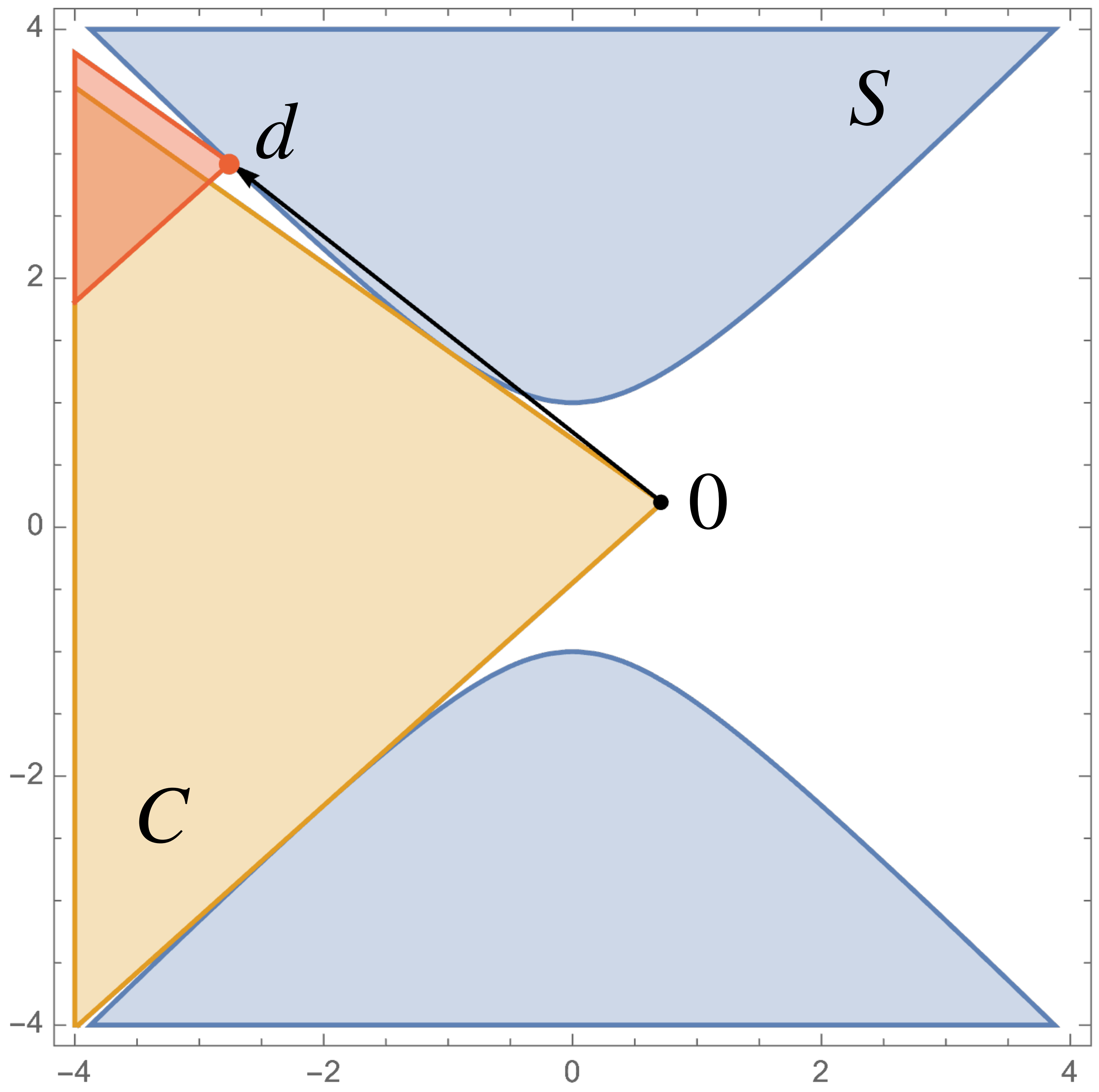

Observation: We can find the monoid by identifying points we can move the apex of C to without intersecting S

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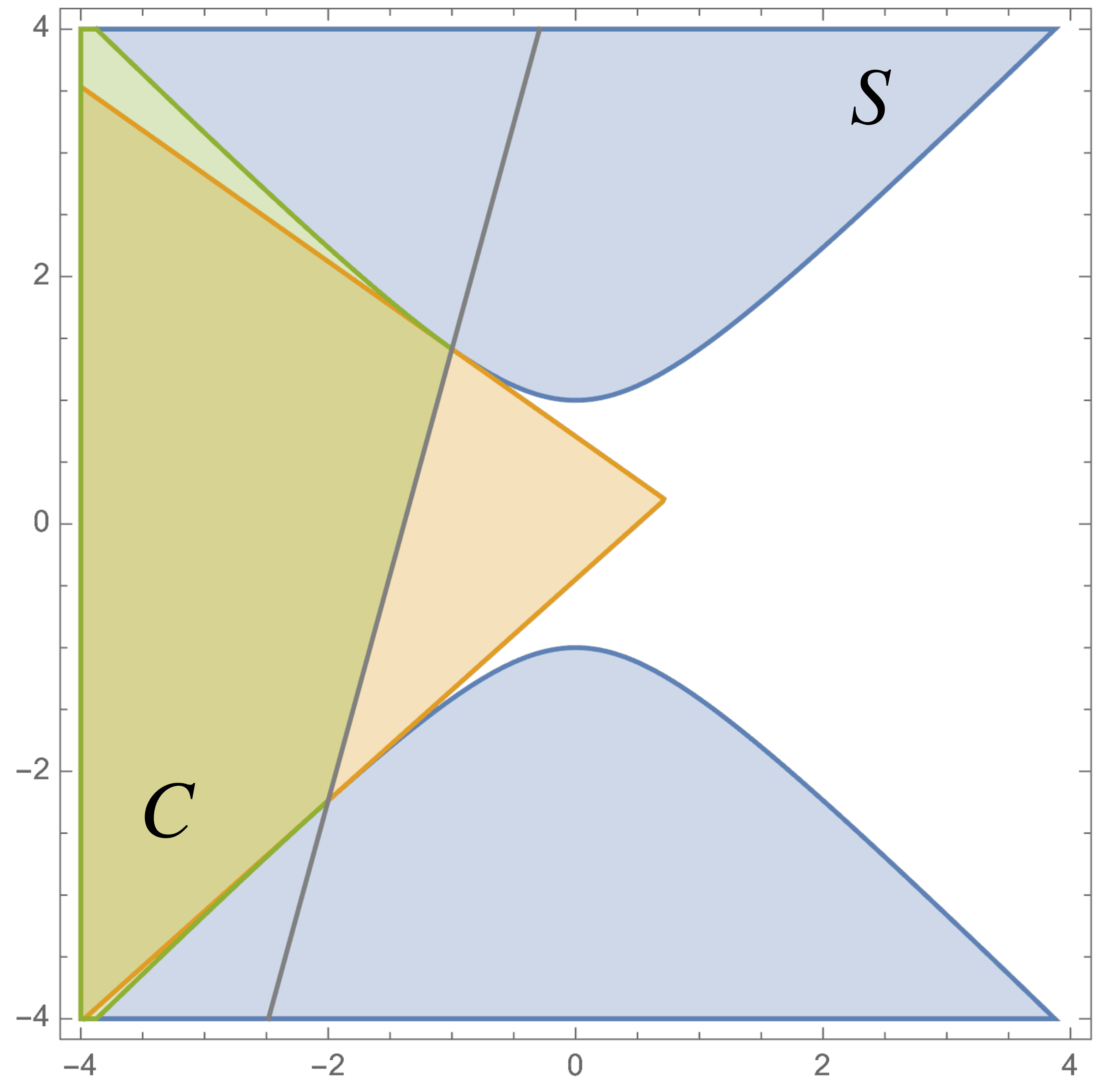

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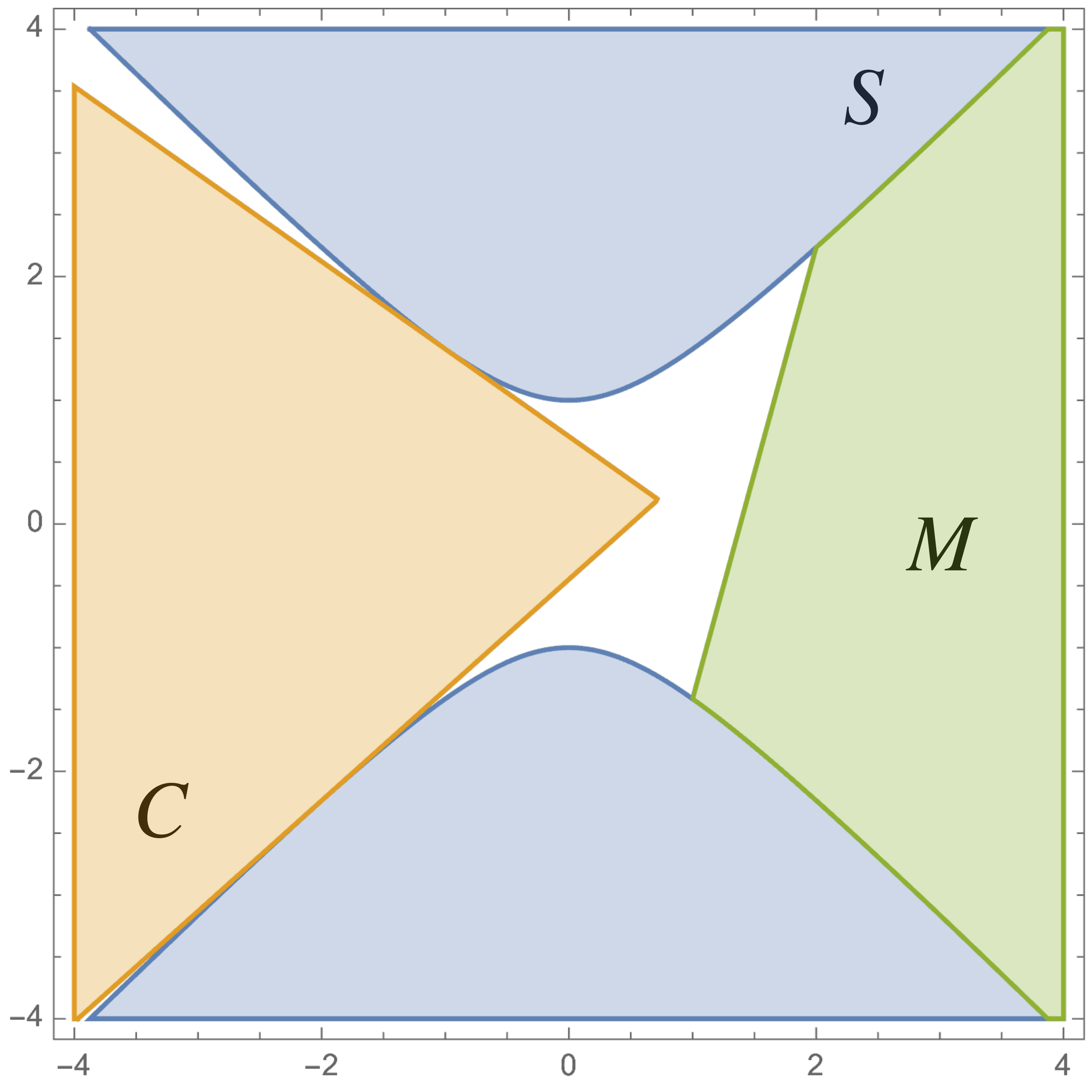

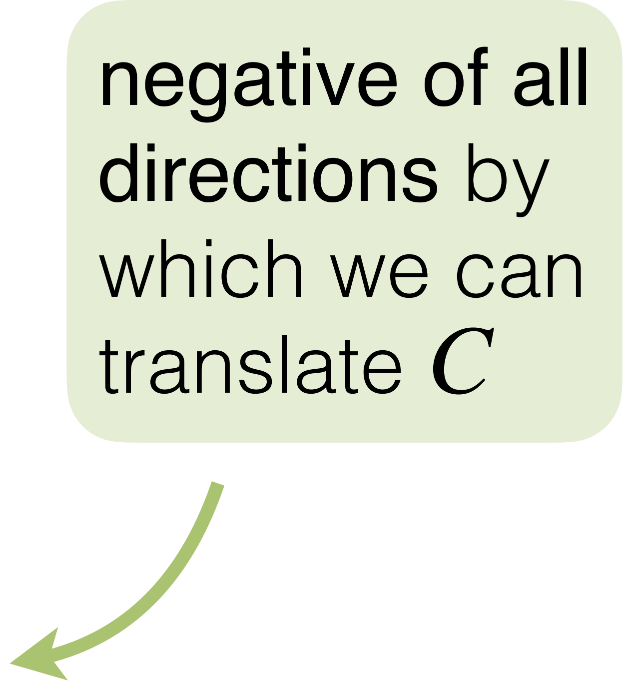

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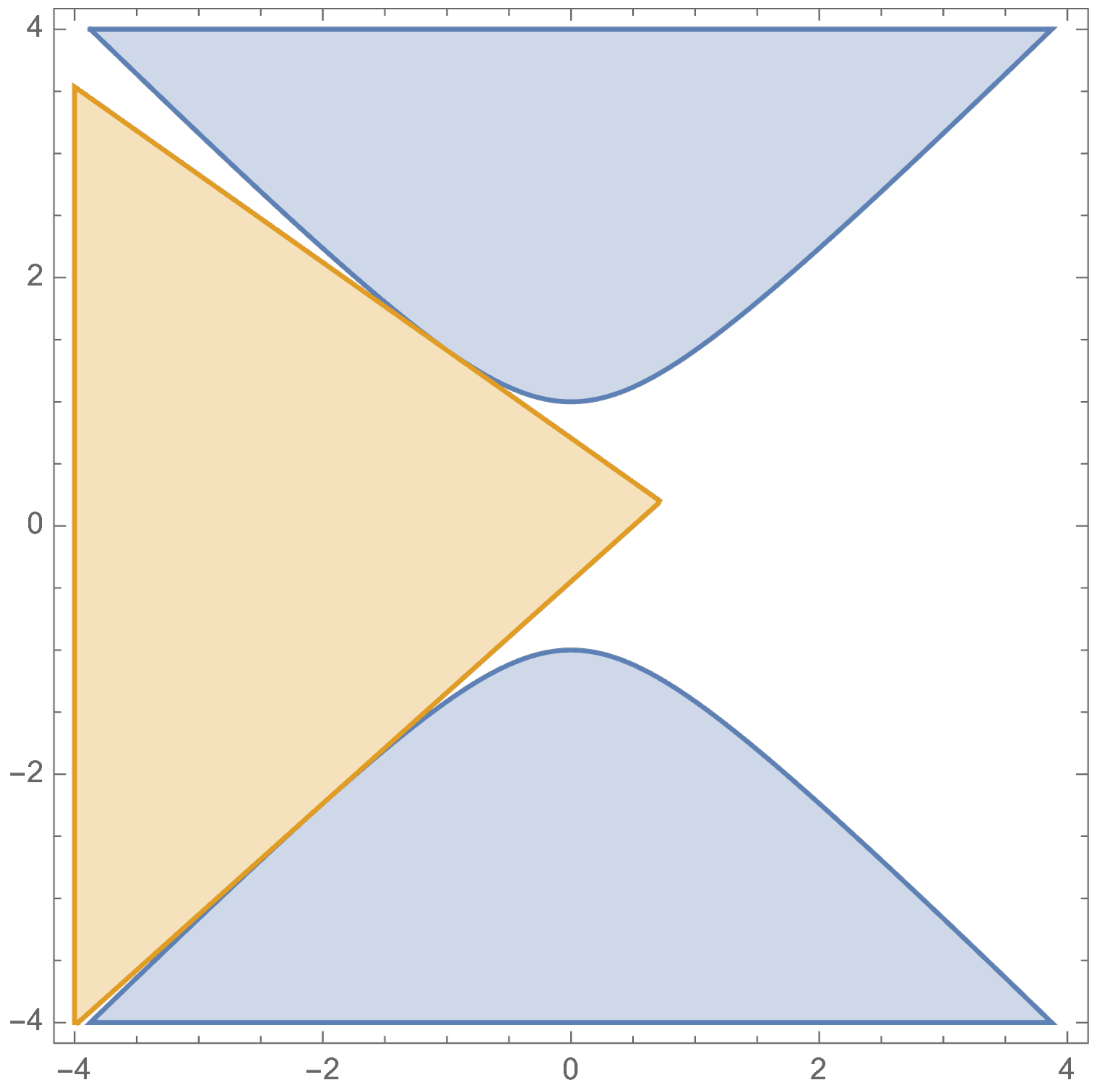
Solving the Monoidal Strengthening Problem

The quadratic case

$$\inf_{m} \quad \phi(r+m)$$

$$\text{s.t.} \quad m \in M$$



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↓

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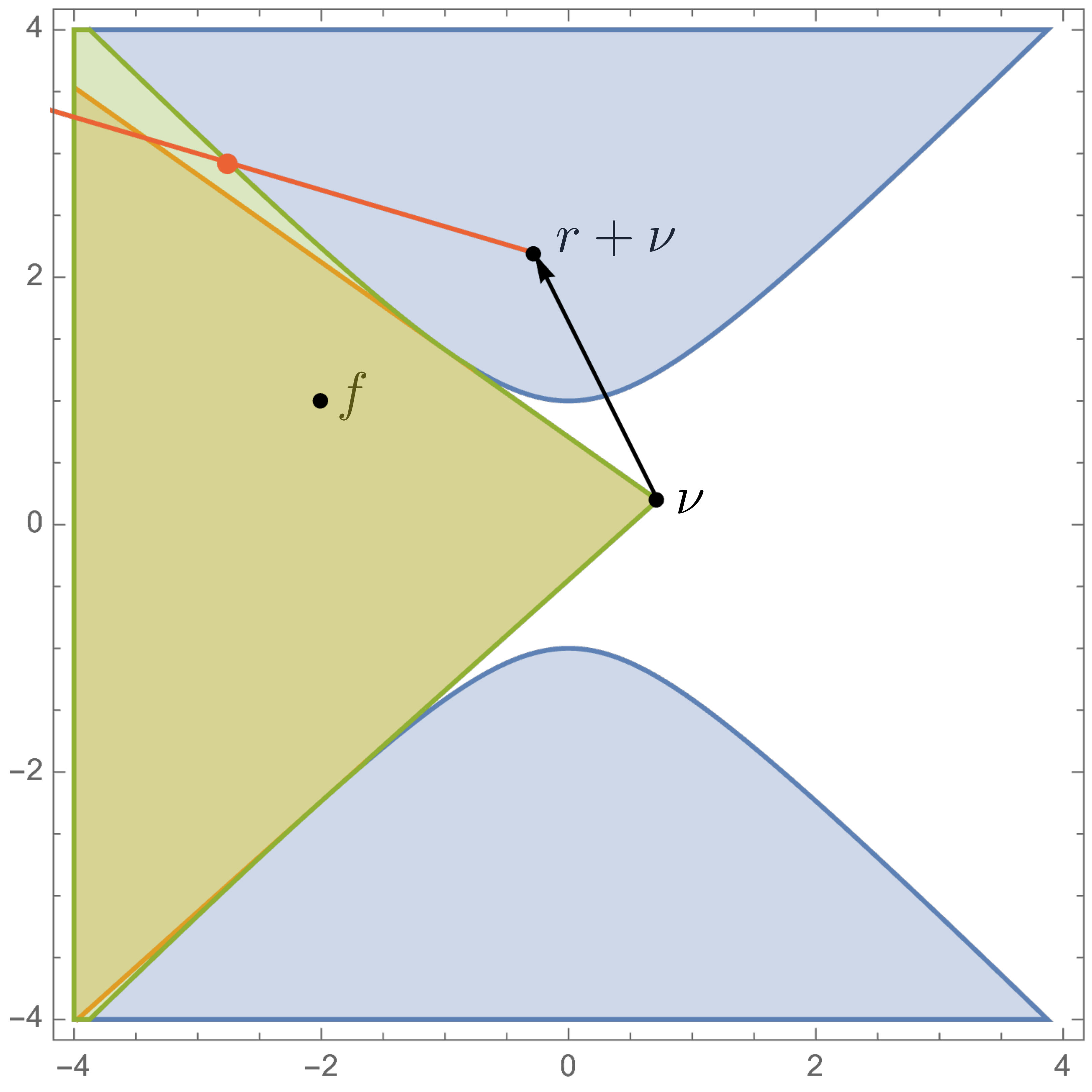

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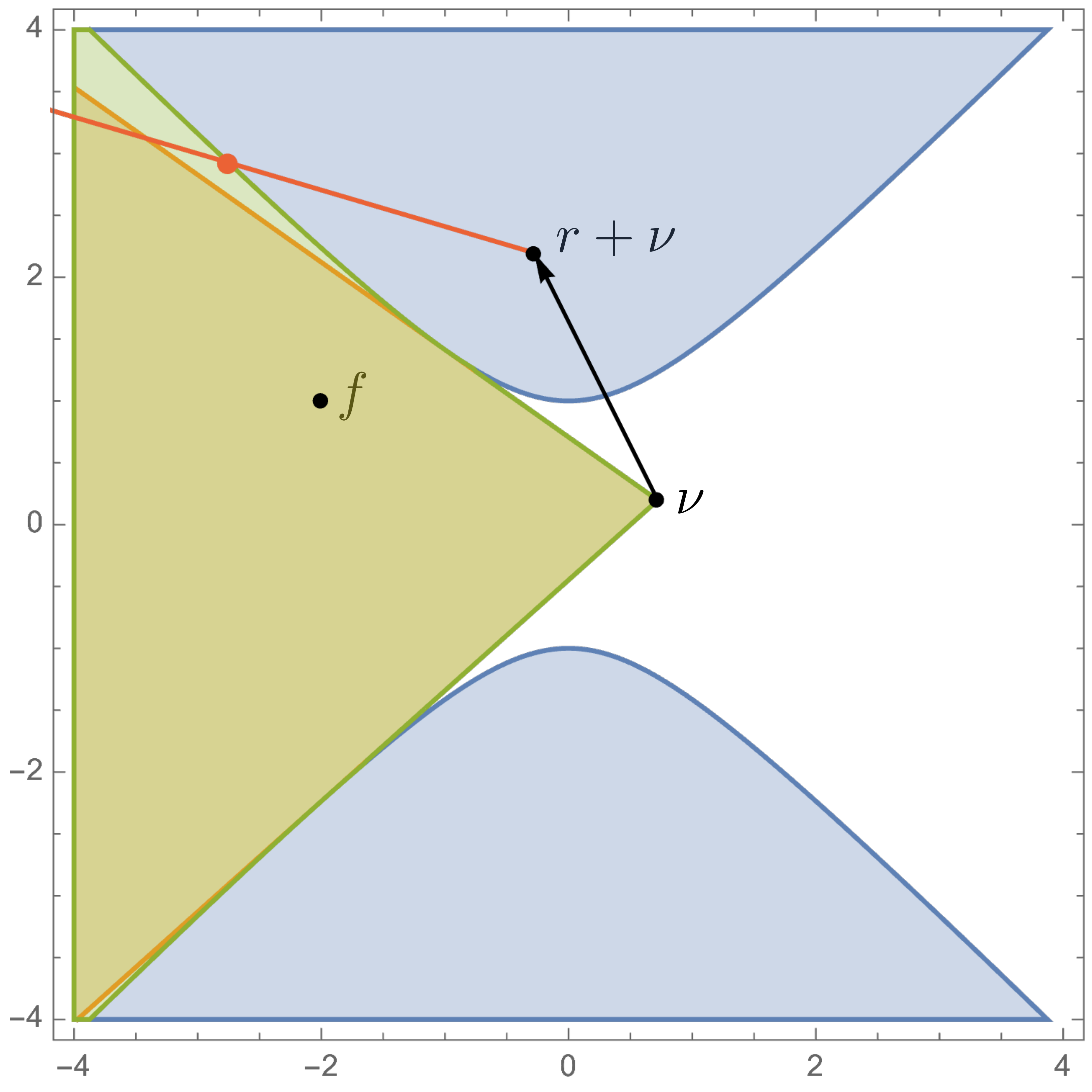
Easy to compute!



Can we do better?

Introduction to the lifting function

Assume $\sum \phi(r^i)x_i + \phi(r)\omega \geq 1$ is a valid cut with $\sum r^i x_i + r\omega \in S$ and $\omega \in \mathbb{Z}_{\geq 1}$.

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$$\implies \pi(r) = \sup_{s,\omega} \frac{1 - \phi(s)}{\omega}$$

$$\text{s.t. } \omega r + s \in S$$

$$\omega \in \mathbb{Z}_{\geq 1}$$

$\pi(r)$ should be as small as possible

$s = \sum r^i x_i$

Solving the Lifting Problem

The quadratic case

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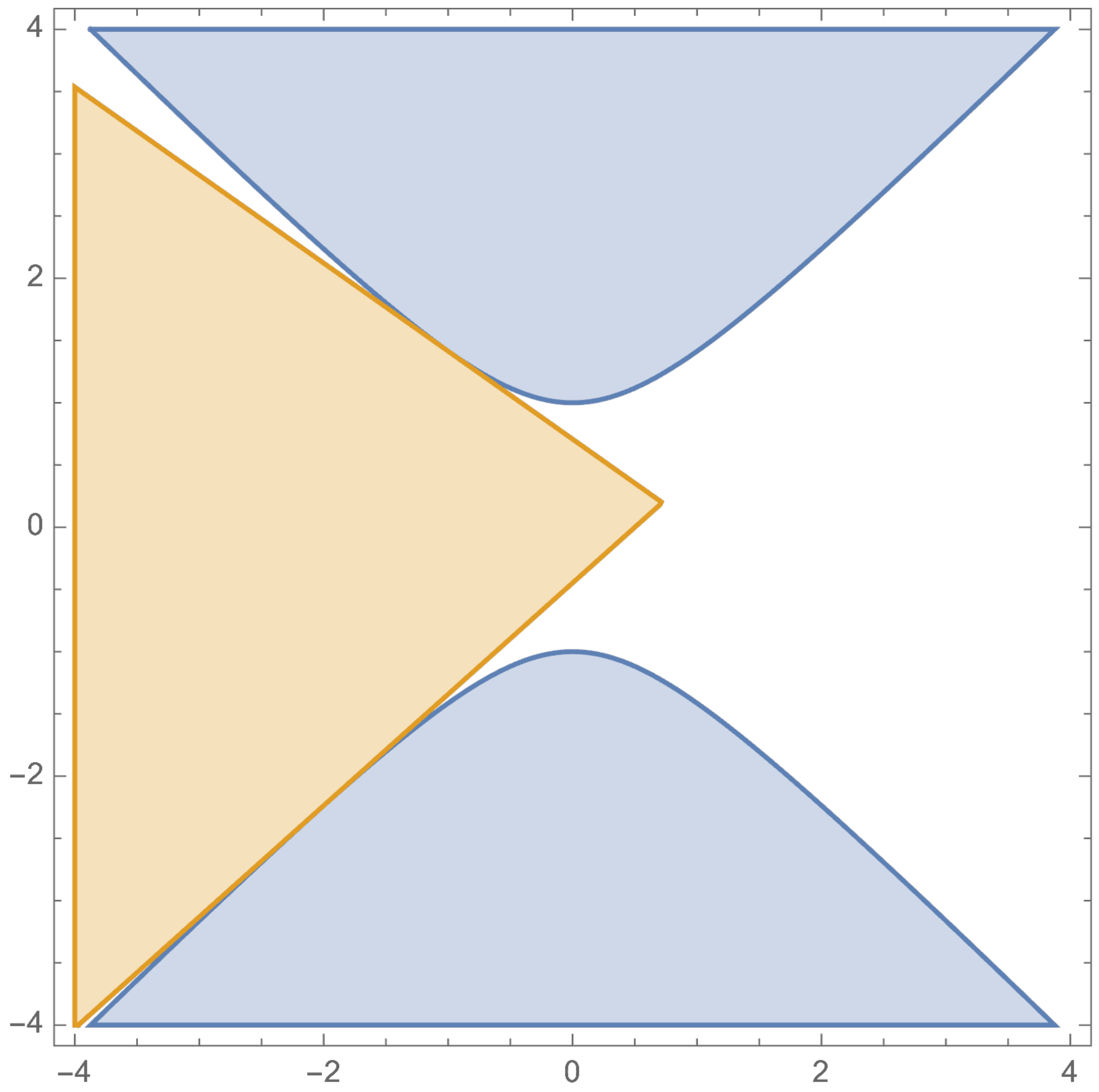

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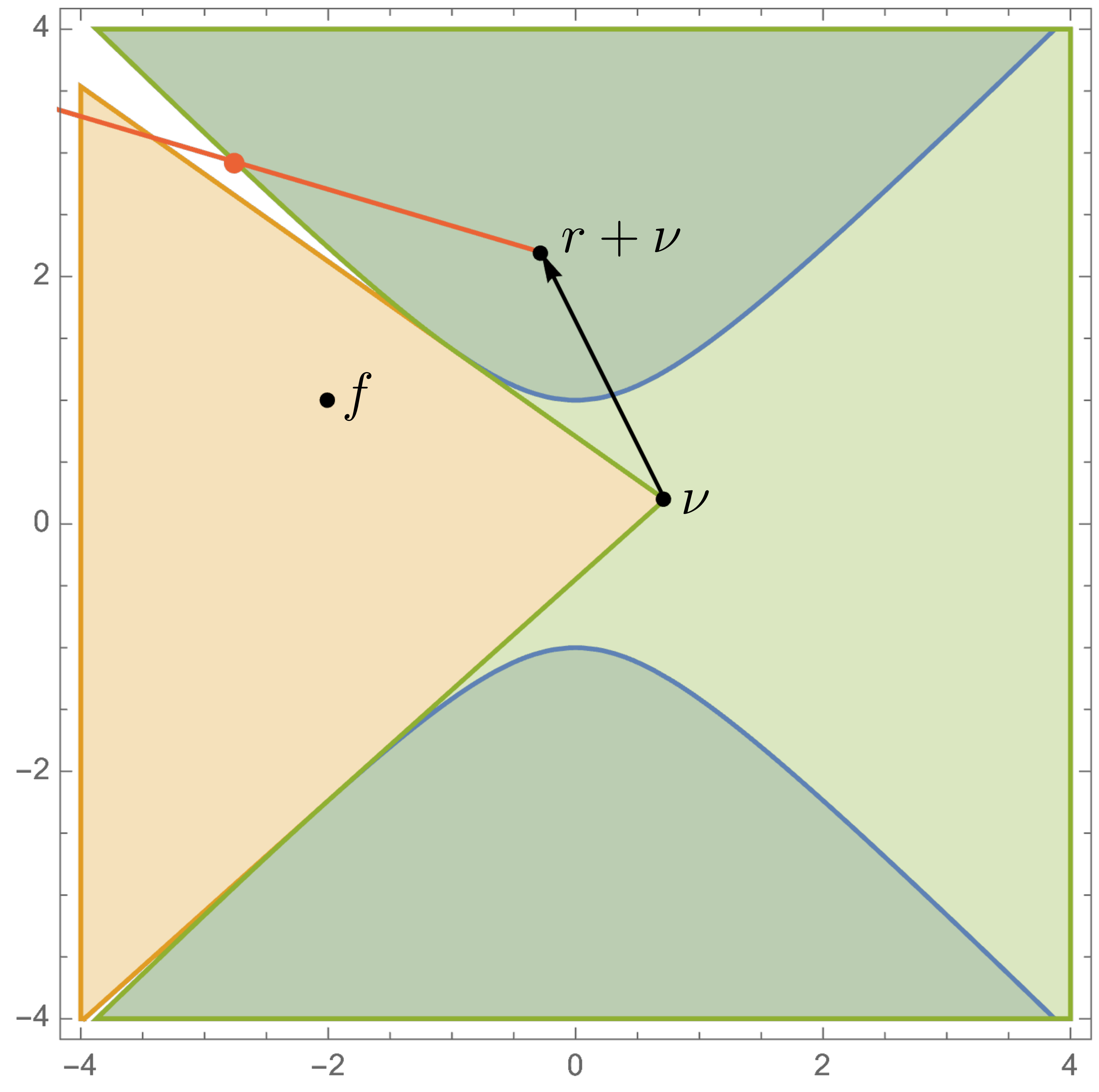
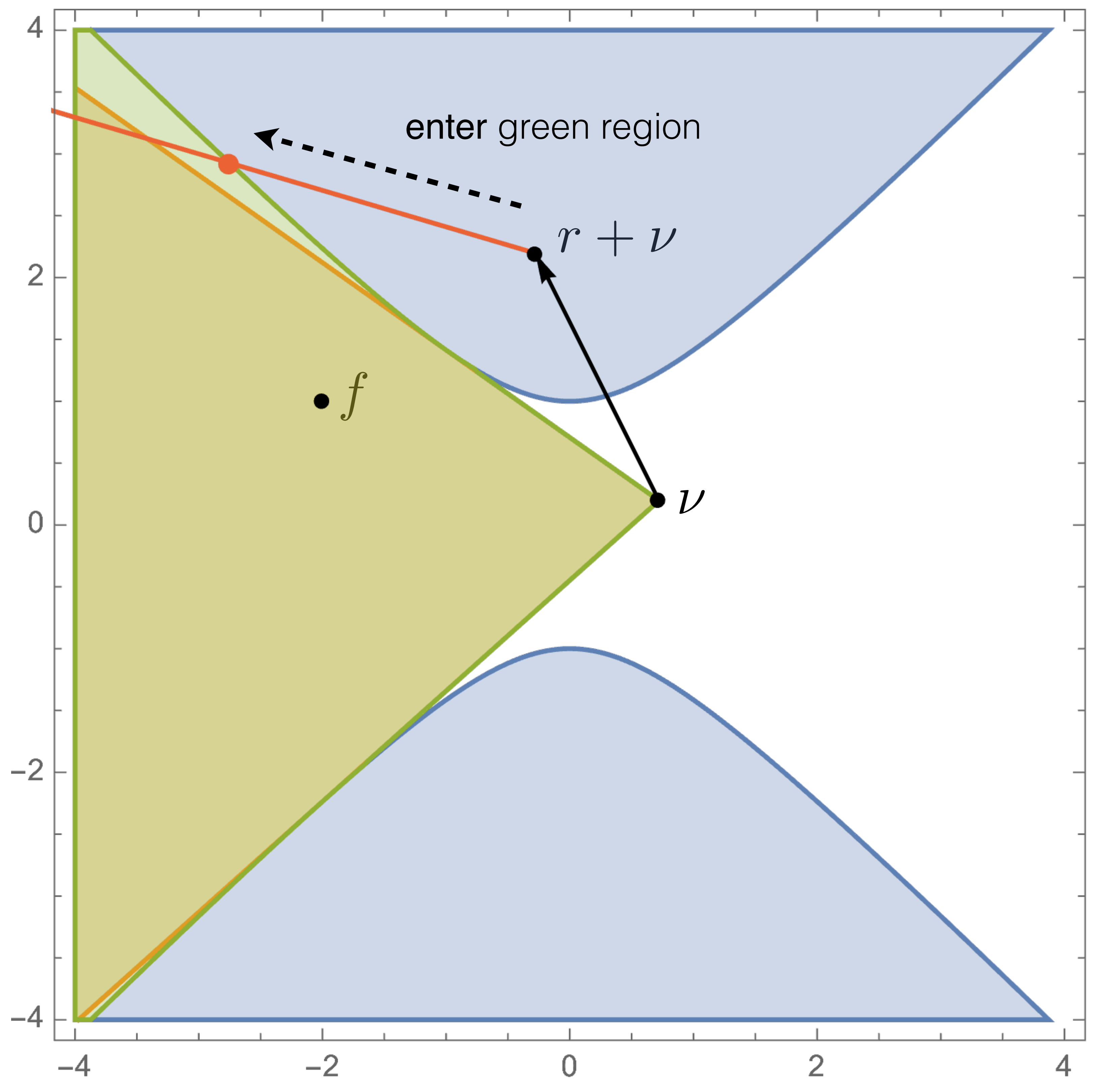
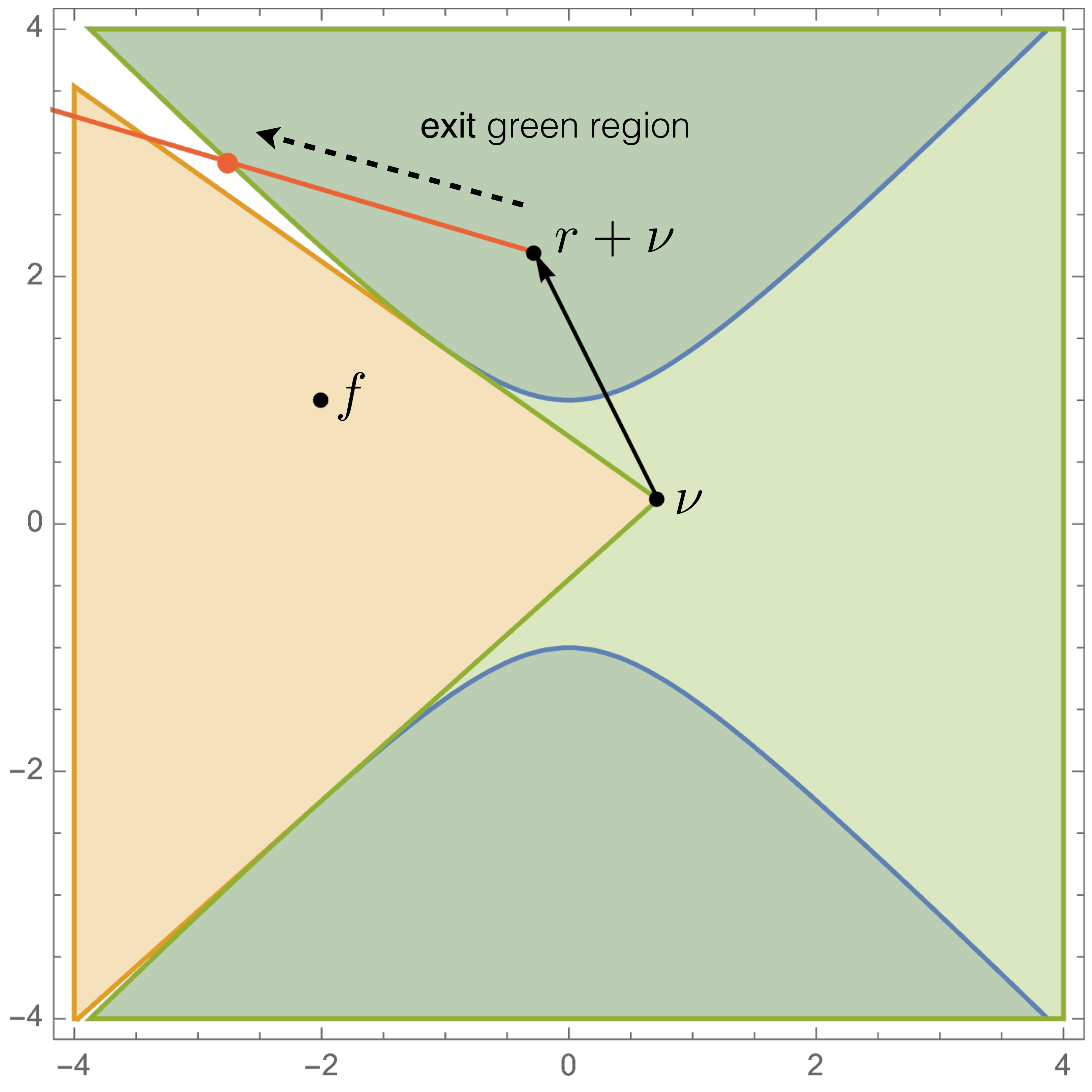


Comparing Monoidal and Lifting

… and realizing that they are the same



Monoidal Strengthening Problem



Lifting Problem

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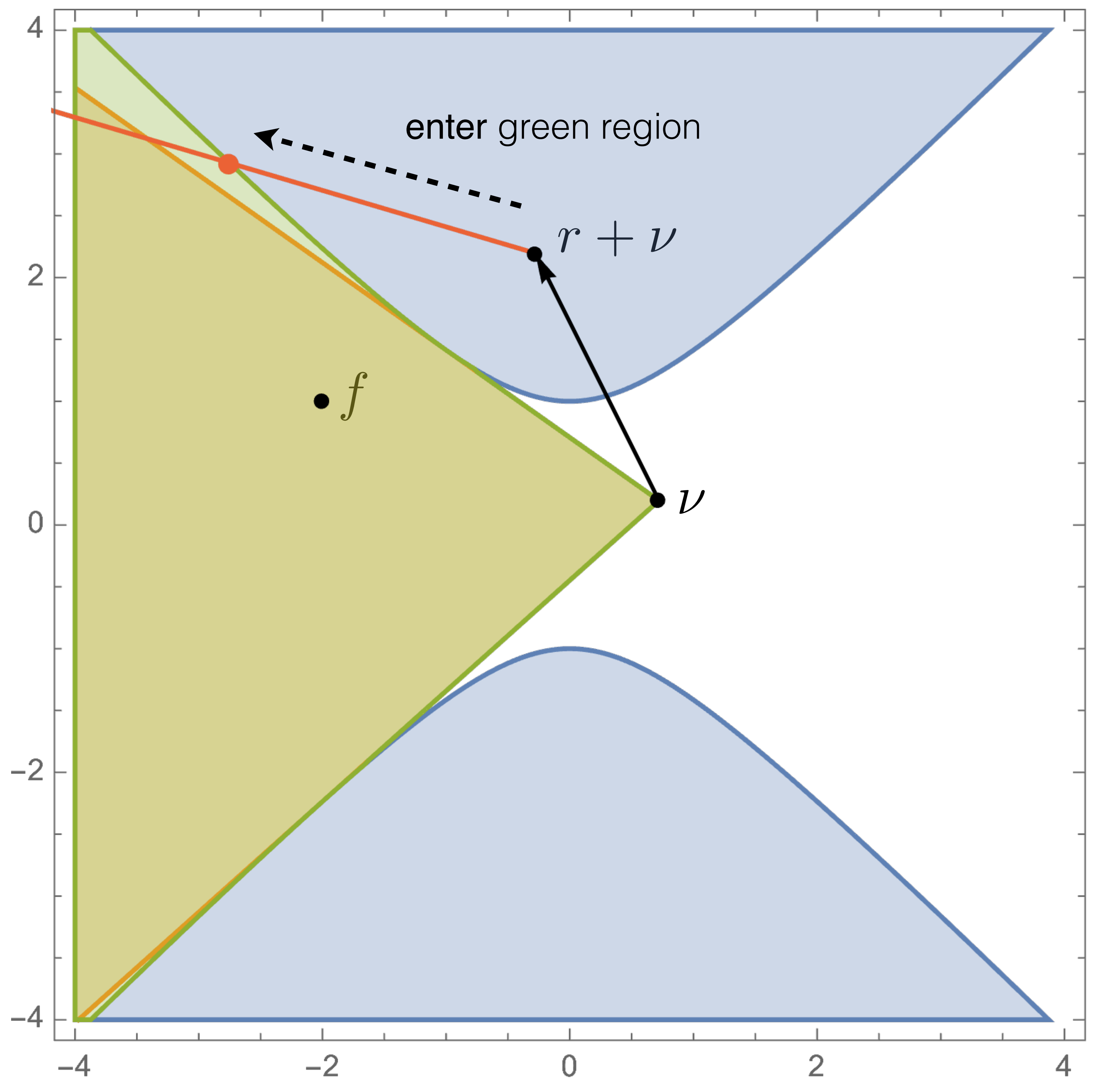
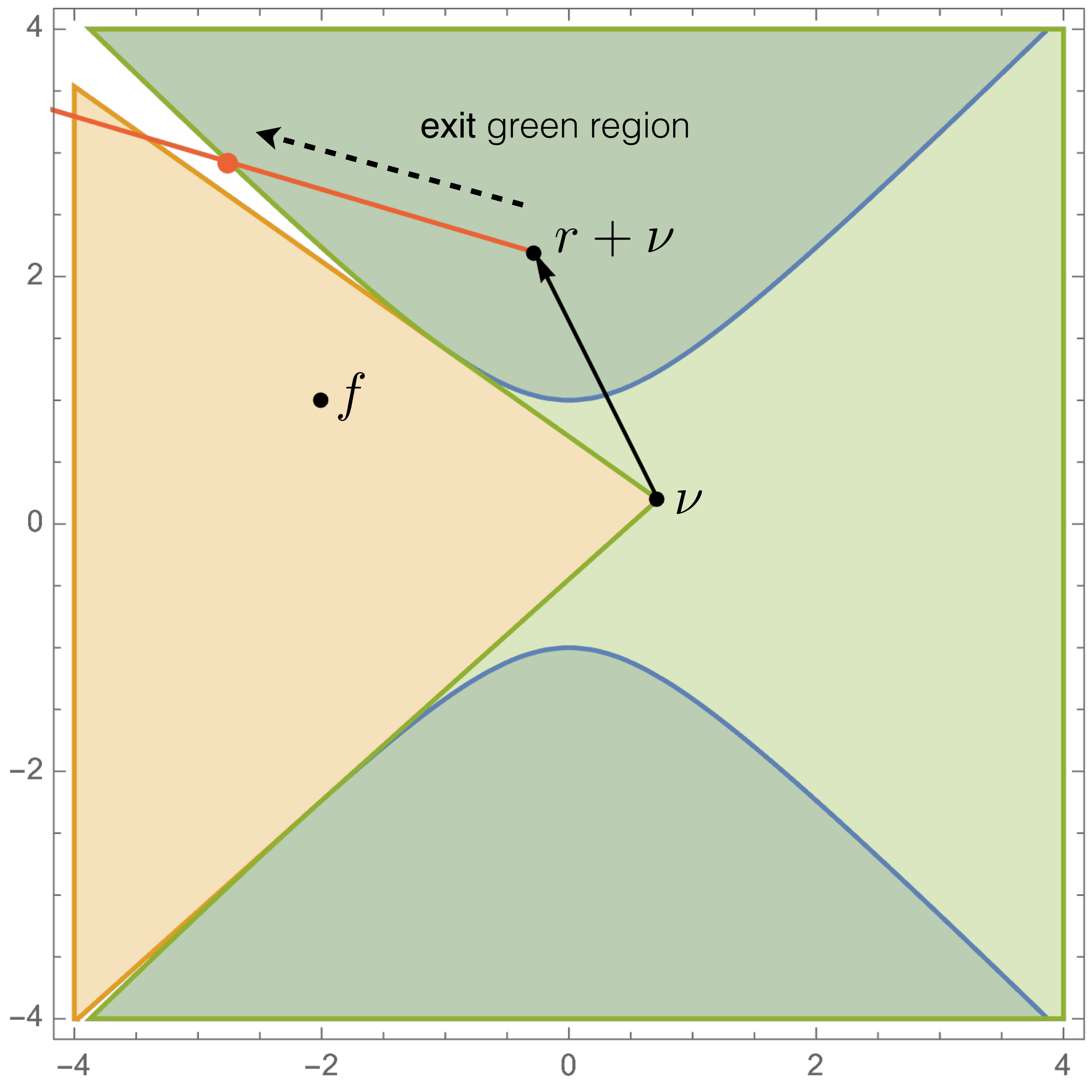


find the **smallest stepsize** so that we **enter** a set

Monoidal Strengthening Problem



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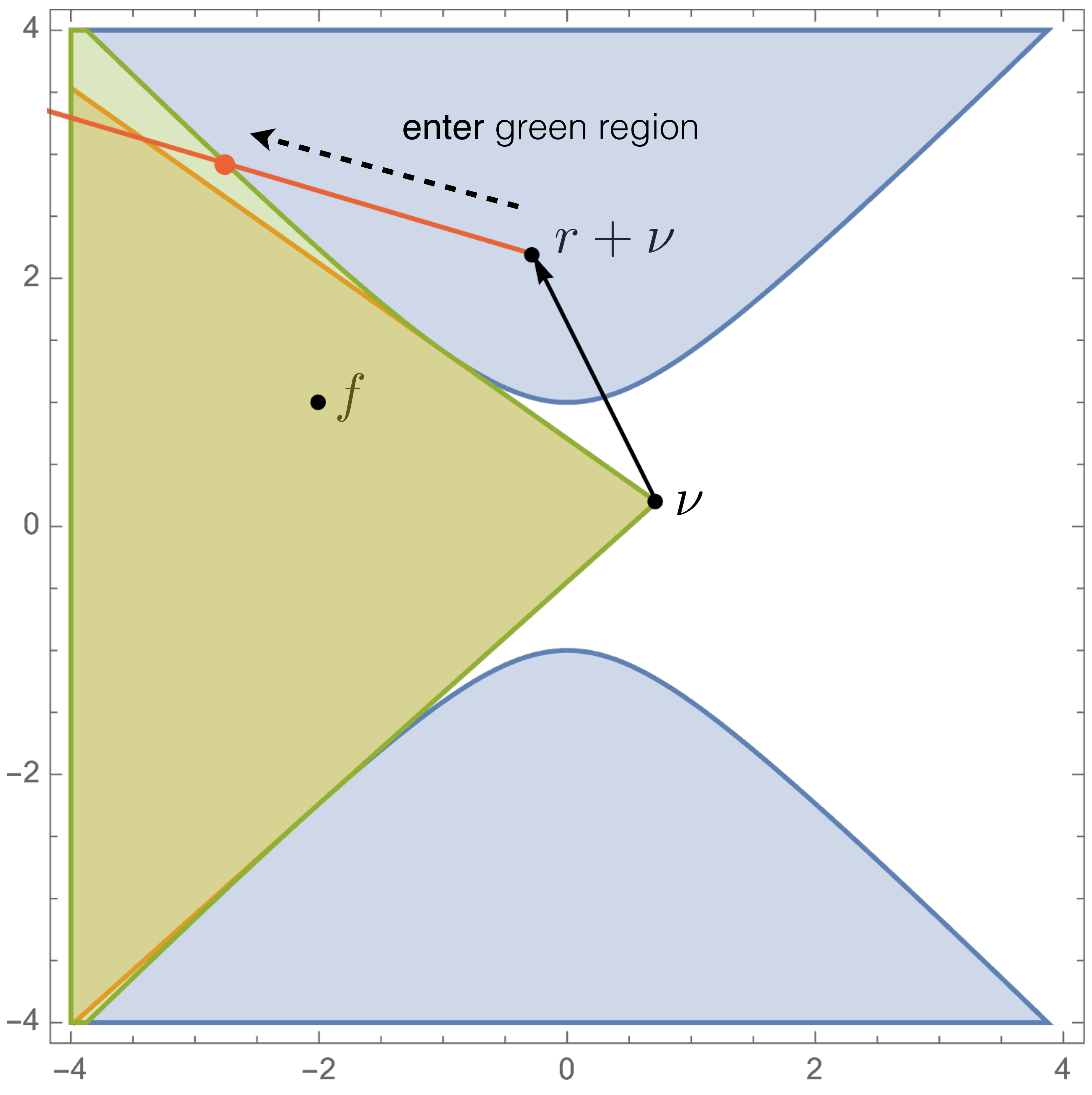

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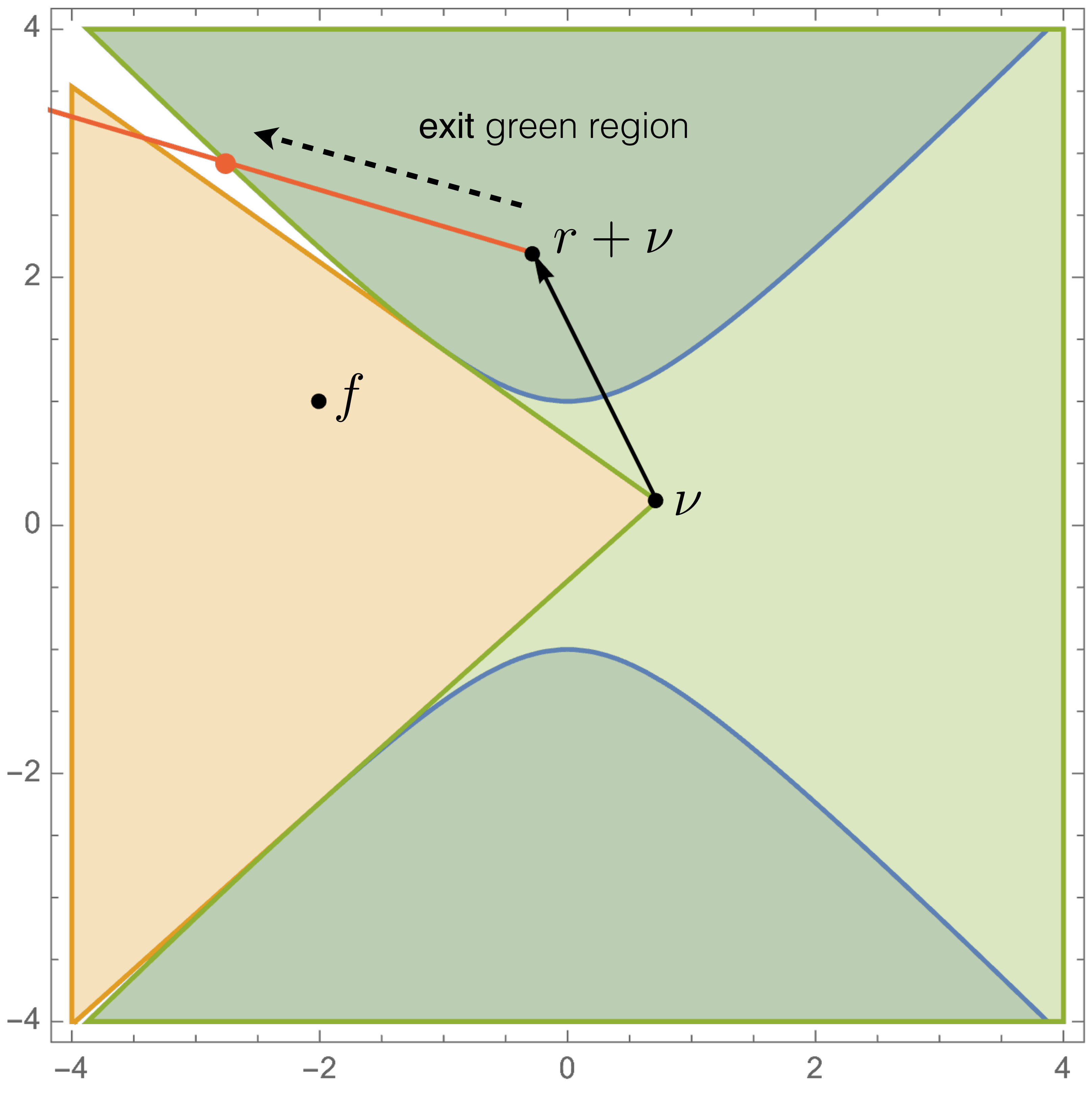
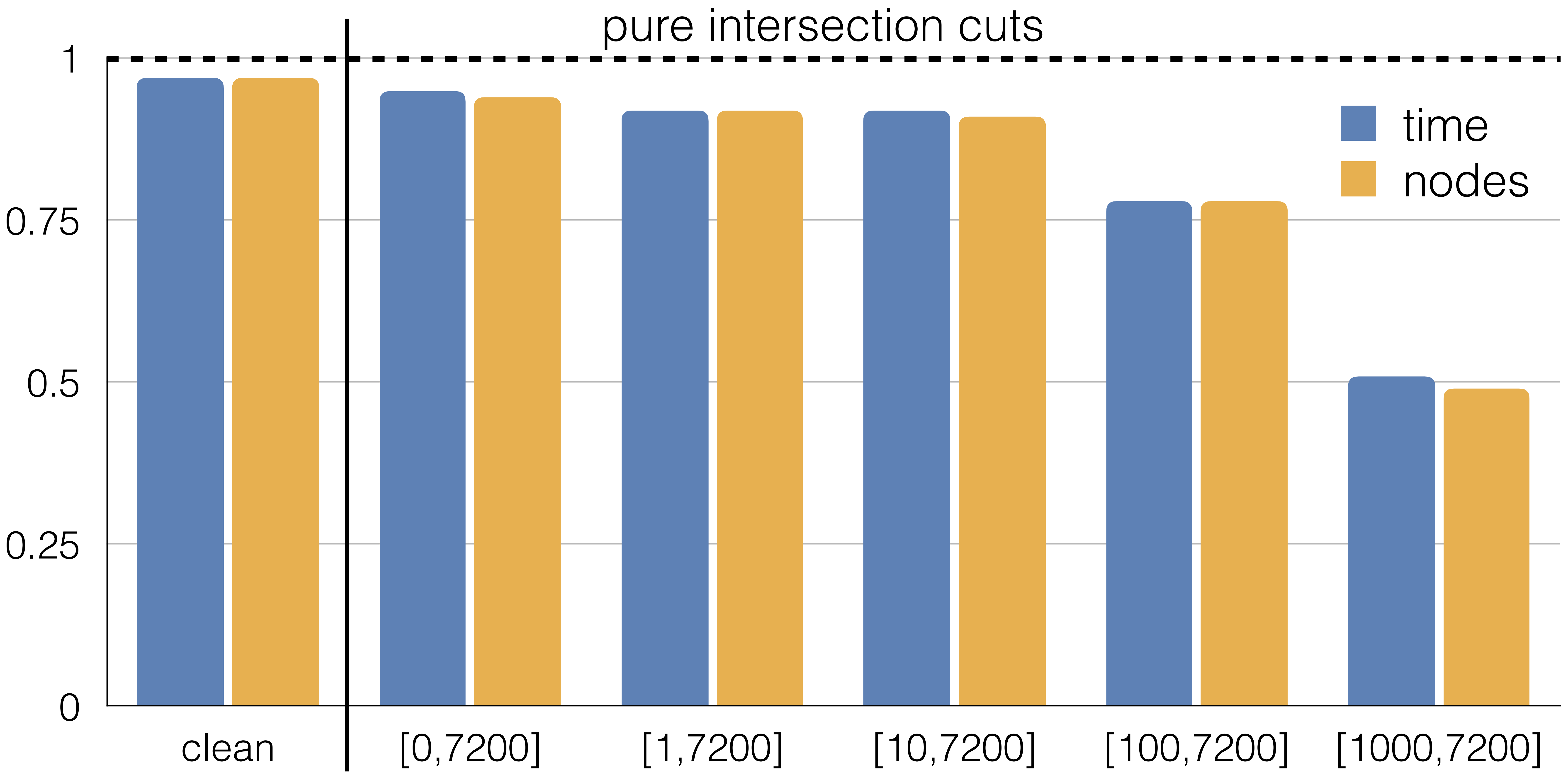


find the **biggest stepsize** so that we **exit** the complement

Lifting Problem

How Does It Perform in Practice?

Branch-and-Bound Experiments



relative shifted geometric mean of B&B using monoidal w.r.t. pure intersection cuts

Contributions

Overview

In this talk:

- We found a suitable monoid for (special) quadratic sets
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Additional contributions:

- We show when monoidal strengthening is actually possible
- We find a minimal representation of C

How Does It Perform in Practice?

Branch-and-Bound Experiments

		pure intersection cuts			*monoidal*			*relative*	
set	instances	solved	time	nodes	solved	time	nodes	time	nodes
clean	189	113	221.87	5282	115	214.63	5321	**0.97**	**0.97**
[0,7200]	115	113	22.81	936	115	21.56	883	**0.95**	**0.94**
[1,7200]	83	81	67.62	2377	83	62.40	2184	**0.92**	**0.92**
[10,7200]	81	79	72.54	2574	81	66.56	2341	**0.92**	**0.91**
[100,7200]	23	21	724.66	186545	23	565.24	144747	**0.78**	**0.78**
[1000,7200]	10	8	2475.04	631764	10	1252.96	307639	**0.51**	**0.49**