An Introduction to Semidefinite Program Relaxations of Quadratically Constrained Quadratic Programs

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    - Examine quadratically constrained quadratic programs (QCQPs) and their semidefinite program (SDPs) relaxations,
    - Understand **structures within QCQPs** that enable us to solve them via SDPs,
    - Exploit structures governing exactness properties to design efficient first-order methods to solve a class of low rank SDPs.

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  - An introduction to QCQPs and their SDP relaxations
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  - Examples

An introduction to SDPs

References:

Ben-Tal, A. and Nemirovski, A. (2001). *Lectures on Modern Convex Optimization*, volume 2 of *MPS-SIAM Ser. Optim.* SIAM

Basic definitions

- $\mathbb{R}^n$ = real column vectors of length n
- $\mathbb{R}^{m\times n}$ = real matrices of size $m \times n$
- $\mathbb{S}^n \subseteq \mathbb{R}^{n\times n}$ = space of $n \times n$ real symmetric matrices

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  $\Longrightarrow$ Symmetry of the matrices ensures that the eigenvalues are *all* real.

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- In $\mathbb{R}^{m \times n}$, we use the *trace (Frobenius) inner product* given by

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Notation

- $\mathbb{S}^n_+$ $=$ set of $n \times n$ positive semidefinite matrices
- $X \in \mathbb{S}^n_+$, or $X \succeq 0$, or X is “PSD"

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- It is *self-dual*, i.e., $\left\{S \in \mathbb{S}^n : \langle S, X\rangle \geq 0, \forall X \in \mathbb{S}^n_+\right\} = \mathbb{S}^n_+$

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  In particular, $X, S \in \mathbb{S}^n_+ \implies \langle S, X\rangle \geq 0$

PSD practice

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- But, symmetry takes away $\binom{n}{2}$ degrees of freedom
- So, its dimension is $\binom{n+1}{2}$

PSD practice

When is a diagonal matrix in $\mathbb{S}^n_+$?

$$D := \begin{pmatrix} D_{11} & 0 & \dots & 0 \\ 0 & D_{22} & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & D_{nn} \end{pmatrix}$$

- D is a diagonal matrix where $\mathrm{diag}(D) = (D_{11}, D_{22}, \dots, D_{nn})^\top$

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- Its eigenvalues are $D_{11}, D_{22}, \dots, D_{nn}$
- So, $D \in \mathbb{S}^n_+$ iff $\mathrm{diag}(D) \geq 0$

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Is the following matrix in $\mathbb{S}^3_+$?

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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- Yes, we can check its principal minors...
- Also, note that it is equal to $vv^\top$ where $v = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

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- Yes, because it is equal to $vv^\top$ where $v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

Theorem

$X \in \mathbb{S}^n_+$ if and only if there exists

- an orthogonal matrix $U \in \mathbb{R}^{n\times n}$, and
- a nonnegative diagonal matrix $D \in \mathbb{S}^n$

such that $X = UDU^\top$.

PSD characterization

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Here, the elements of $\mathrm{diag}(D)$ are precisely the eigenvalues of X, and the columns of U are the corresponding eigenvectors of X.

A semidefinite program

Primal SDP problem:

$$\mathrm{Opt}(P) := \inf_{X \in \mathbb{S}^n} \left\{ \langle C, X \rangle : \begin{array}{ll} \langle A_i, X \rangle = b_i, & \forall i \in [m], \\ X \succeq 0 & \end{array} \right\},$$

where

- the decision variable is $X \in \mathbb{S}^n$
- the data are the matrices $C, A_1, \ldots, A_m \in \mathbb{S}^n$, and the vector $b \in \mathbb{R}^m$

SDP practice

Specify the data for this problem:

$$\inf_{X \in \mathbb{S}^2} \left\{ X_{12} : \begin{array}{c} X_{11} + X_{22} = 1 \\ \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq 0 \end{array} \right\}$$

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- $n = 2$ and $m = 1$
- $C = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $b_1 = 1$

SDP practice

What is the optimum value of this problem?

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$$\text{Opt}^* = \inf_{X_{11}, X_{22}, X_{12}} \left\{ X_{12} : \begin{array}{c} X_{11} + X_{22} = 1 \\ X_{11} \geq 0, \ X_{22} \geq 0 \\ X_{12}^2 \leq X_{11} X_{22} \end{array} \right\}$$

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$$= \inf_{X_{11}} \left\{ -\sqrt{X_{11}(1 - X_{11})} : \ 0 \leq X_{11} \leq 1 \right\}$$

$$\text{Opt}^* = -\frac{1}{2} \quad \text{and} \quad X^* = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

What can be expressed as an SDP?

LP is a special case of SDP:

$$
\begin{aligned}
& \inf_{x \in \mathbb{R}^n} \left\{ \langle c, x \rangle : \begin{array}{ll} \langle a_i, x \rangle = b_i, & \forall i \in [m], \\ x \geq 0 & \end{array} \right\} \\
\iff & \inf_{X \in \mathbb{S}^n} \left\{ \langle \mathrm{Diag}(c), X \rangle : \begin{array}{ll} \langle \mathrm{Diag}(a_i), X \rangle = b_i, & \forall i \in [m], \\ X \succeq 0 & \end{array} \right\}
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$$

What can be expressed as an SDP?

Second-order cone programs (SOCPs) are a special case of SDPs:

$$\|x\|_2 \leq t \quad \Longleftrightarrow \quad \begin{pmatrix} t & x^\top \\ x & tI_n \end{pmatrix} \succeq 0$$

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This is based on the following very useful result:

Theorem (Schur Complement Lemma)

Consider a symmetric matrix $M := \begin{pmatrix} P & Q^\top \\ Q & R \end{pmatrix}$ such that R is positive definite. Then, $M \succeq 0$ iff $P - Q^\top R^{-1} Q \succeq 0$.

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Many nice functions of eigenvalues (or singular values) of matrices admit SDP representations. . .

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Theorem

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex, SDP representable, permutation invariant function, i.e., $f(x) = f(Px)$ for every permutation matrix P.
- Let $\lambda(X)$ denote the vector of eigenvalues of matrix $X \in \mathbb{S}^n$.

Then, the epigraph of the function $F(X) = f(\lambda(X)) : \mathbb{S}^n \to \mathbb{R}$ admits an SDP representation.

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- $\lambda_{\max}(X)$, $\sum_{i\in[n]} \lambda_i(X)$,
- $\|X\|_p := \|\lambda(X)\|_p = \left(\sum_{i\in[n]} |\lambda_i(X)|^p\right)^{1/p}$ for $p \in \mathbb{Q}$ and $p \geq 1$,
- $-\log\det(X) = -\sum_{i\in[n]} \log(\lambda_i(X))$ for $X \succ 0$, ...

Conic problems and their duals

Consider the conic optimization problem is

$$\mathrm{Opt}(P) := \inf_X \left\{ \langle C, X \rangle : \begin{array}{ll} \langle A_i, X \rangle = b_i, & \forall i \in [m], \\ X \in \mathbb{K} & \end{array} \right\}.$$

where $\mathbb{K}$ is a proper cone.

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Given a cone $\mathbb{K}$, define the dual cone as

$$\mathbb{K}_* := \{\xi : \ \langle \xi, X \rangle \geq 0, \ \forall X \in \mathbb{K}\}.$$

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Then, the dual conic problem is given by

$$\mathrm{Opt}(D) := \sup_{y \in \mathbb{R}^m, S} \left\{ \langle b, y \rangle : \begin{array}{l} \sum_{i \in [m]} A_i y_i + S = C, \\ S \in \mathbb{K}_* \end{array} \right\}.$$

Conic duality

Theorem (Weak Duality Theorem)

- Let (P) and (D) be any pair of primal and dual conic programs, where the primal (P) is in minimization form.
- Let $\bar{X}$ be a primal feasible solution, and $(\bar{y}, \bar{S})$ be a dual feasible solution.

Then,

$$\left\langle C, \bar{X} \right\rangle - \langle b, \bar{y} \rangle = \left\langle \bar{S}, \bar{X} \right\rangle \geq 0.$$

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Proof.

Conic duality

Corollary (Weak Duality Theorem)

Let $\bar{X}$ be a primal feasible solution to (P) (in minimization form), and $(\bar{y}, \bar{S})$ be a dual feasible solution to its dual (D). Then,

$$\langle C, \bar{X} \rangle \geq \mathrm{Opt}(P) \geq \mathrm{Opt}(D) \geq \langle b, \bar{y} \rangle .$$

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Corollary

Let $\bar{X}$ be a primal feasible solution to (P) (in minimization form), and $(\bar{y}, \bar{S})$ be a dual feasible solution to its dual (D).

If $\left\langle C, \bar{X} \right\rangle = \langle b, \bar{y} \rangle$, then $\bar{X}$ is primal optimum and $(\bar{y}, \bar{S})$ is dual optimum.

Moreover, in the case of SDPs, $\left\langle \bar{X}, \bar{S} \right\rangle = 0$ iff $\bar{X}\bar{S} = 0$.

Dual SDP

Recall our primal SDP:

$$\mathrm{Opt}(P) := \inf_{X \in \mathbb{S}^n} \left\{ \langle C, X \rangle : \begin{array}{ll} \langle A_i, X \rangle = b_i, & \forall i \in [m], \\ X \succeq 0 & \end{array} \right\}.$$

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Then, the dual SDP is given by

$$\begin{aligned} \mathrm{Opt}(D) &:= \sup_{y \in \mathbb{R}^m, S \in \mathbb{S}^n} \left\{ \langle b, y \rangle : \begin{array}{l} \sum_{i \in [m]} A_i y_i + S = C, \\ S \succeq 0 \end{array} \right\} \\ &= \sup_{y \in \mathbb{R}^m} \left\{ \langle b, y \rangle : C - \sum_{i \in [m]} A_i y_i \succeq 0 \right\}. \end{aligned}$$

SDP practice example

What is the dual of the following SDP?

$$\inf_{X \in \mathbb{S}^2} \left\{ X_{12} : \begin{array}{c} X_{11} + X_{22} = 1 \\ \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq 0 \end{array} \right\}$$

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- $\text{Opt}(D) = -\frac{1}{2}$
- $y_1^* = -\frac{1}{2}$ and $S^* = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

SDP practice example

Let's verify...

$$X^* = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad S^* = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

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$$X^* S^* = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Strong duality?

Do we always have strong duality, i.e., $\mathrm{Opt}(P) = \mathrm{Opt}(D)$?

SDP strong duality counter example

Consider

$$\mathrm{Opt}(P) = \inf_{X \in \mathbb{S}^3} \left\{ X_{11} : \begin{array}{l} X_{22} = 0, \\ X_{11} + 2X_{23} = 1, \\ X \succeq 0 \end{array} \right\}$$

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- $\mathrm{Opt}(P) = 1$ while $\mathrm{Opt}(D) = 0$.
- Positive and finite duality gap ?!?!?!

Theorem (Strong conic duality)

Let (P) and (D) be a pair of feasible primal and dual conic programs, where the primal (P) is in minimization form.

- If $\exists$ a primal feasible $\bar{X}$ with $\bar{X} \in \operatorname{int}(\mathbb{K})$ (i.e., primal strict feas. holds), then $\operatorname{Opt}(P) = \operatorname{Opt}(D)$ and $\operatorname{Opt}(D)$ is attained.
- If $\exists$ a dual feasible $(\bar{y}, \bar{S})$ with $\bar{S} \in \operatorname{int}(\mathbb{K})$ (i.e., dual strict feas. holds), then $\operatorname{Opt}(P) = \operatorname{Opt}(D)$ and $\operatorname{Opt}(P)$ is attained.
- If both primal and dual strict feas. hold, then $\exists$ primal-dual optimal solutions $(\bar{X}, \bar{y}, \bar{S})$ s.t.

$$\operatorname{Opt}(P) = \langle C, \bar{X} \rangle = \langle b, \bar{y} \rangle = \operatorname{Opt}(D) \quad \text{(and for SDPs } \bar{X}\bar{S} = 0\text{)}.$$

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Notation

- $\mathbb{S}^n_{++}$ = set of $n \times n$ positive definite matrices
- $X \in \mathbb{S}^n_{++}$, or $X \succ 0$, or X is "PD"

SDP in practice

Remark

- Be careful about strict feasibility and attainment conditions when applying conic duality!
- Papers (especially the ones focusing on algorithms) often assume that both (P) and (D) have nonempty interior. But, it is best to double check in any given application!

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- More on solving SDPs tomorrow. . .

An introduction to QCQPs

Quadratically constrained quadratic programs (QCQPs)

- $q_{\mathsf{obj}}, q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ quadratic (possibly nonconvex!)

$$\mathrm{Opt} := \inf_{x \in \mathbb{R}^n} \left\{ q_{\mathsf{obj}}(x) : \; q_i(x) \leq 0, \; \forall i \in [m] \right\}$$

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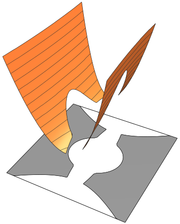

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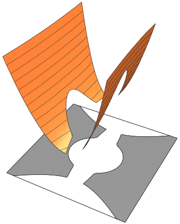

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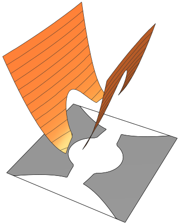

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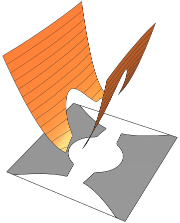

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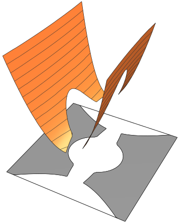

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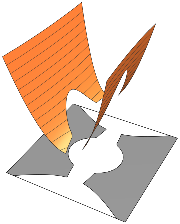

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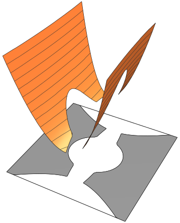

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  [Laurent and Poljak, 1995]
- Interested in sufficient (and perhaps also necessary) conditions for **SDP exactness.**

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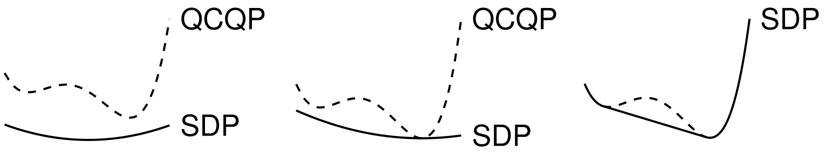

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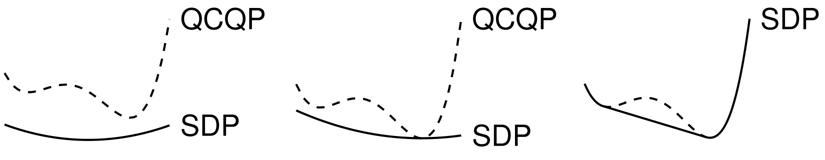

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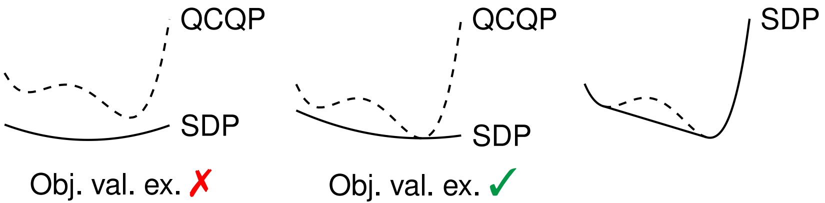

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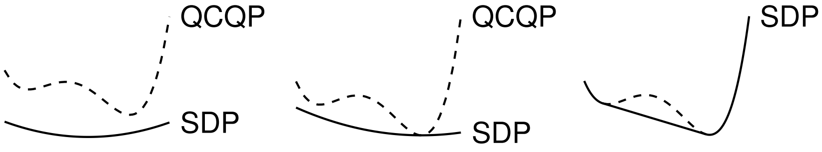

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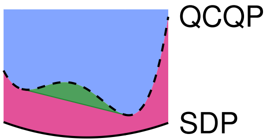

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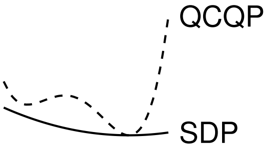

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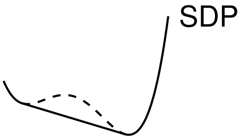

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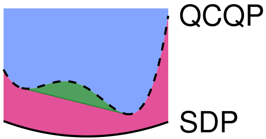

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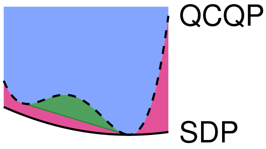

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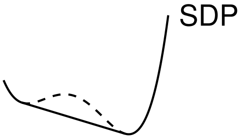

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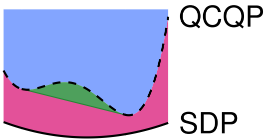

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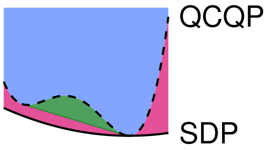

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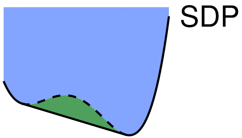

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    $\longrightarrow$ exactness in the lifted SDP space

Exactness in the lifted SDP space: ROG property

References:

Argue, C., K.-K., F., and Wang, A. L. (2022+). Necessary and sufficient conditions for rank-one generated cones. *Math. Oper. Res.*, Forthcoming, (arXiv:2007.07433)

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ROG

- Given $\mathcal{M} \subseteq \mathbb{S}^{n+1}$, define $\mathcal{S}(\mathcal{M}) := \left\{ Z \in \mathbb{S}^{n+1}_+ : \langle M, Z \rangle \leq 0, \ \forall M \in \mathcal{M} \right\}$

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Definition

A closed cone $\mathcal{S} \subseteq \mathbb{S}_+^{n+1}$ is rank-one generated (ROG) if

$$\mathcal{S} = \operatorname{conv}\left(\mathcal{S} \cap \left\{ zz^\top : z \in \mathbb{R}^{n+1} \right\} \right).$$

Equivalently, if all extreme rays are generated by rank-one matrices.

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- **Analogy:** (Integer programs, integral polyhedra) $\approx$ (QCQPs, ROG)

Motivation: ROG $\Longrightarrow$ exactness

For any $\mathcal{M} \subseteq \mathbb{S}^{n+1}$, $\quad \mathrm{Opt} = \inf_{x \in \mathbb{R}^n} \left\{ \binom{x}{1}^\top M_{\mathsf{obj}} \binom{x}{1} : \ \binom{x}{1}\binom{x}{1}^\top \in \mathcal{S}(\mathcal{M}) \right\}$

$$\geq \inf_{Z \in \mathbb{S}^{n+1}} \left\{ \langle M_{\mathsf{obj}}, Z \rangle : \begin{array}{l} Z \in \mathcal{S}(\mathcal{M}) \\ \langle e_{n+1} e_{n+1}^\top, Z \rangle = 1 \end{array} \right\} = \mathrm{Opt}_{\mathsf{SDP}} .$$

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- $\mathcal{S}(\mathcal{M})$ is ROG $\implies$ objective value exactness.

Proposition

- $\mathcal{S}(\mathcal{M})$ is ROG iff for all $M_{\text{obj}} \in \mathbb{S}^{n+1}$,

$$\inf_{z\in\mathbb{R}^{n+1}} \left\{ \left\langle M_{\text{obj}}, zz^\top \right\rangle : zz^\top \in \mathcal{S}(\mathcal{M}) \right\} = \inf_{Z\in\mathbb{S}^{n+1}} \left\{ \langle M_{\text{obj}}, Z\rangle : Z \in \mathcal{S}(\mathcal{M}) \right\}.$$

- If $\mathcal{S}(\mathcal{M})$ is ROG, then for all $B, M_{\text{obj}} \in \mathbb{S}^{n+1}$ s.t. $\mathrm{Opt}_{\text{SDP}} > -\infty$,

$$\inf_{z\in\mathbb{R}^{n+1}} \left\{ \left\langle M_{\text{obj}}, zz^\top \right\rangle : \begin{array}{l} zz^\top \in \mathcal{S}(\mathcal{M}) \\ \left\langle B, zz^\top \right\rangle = 1 \end{array} \right\} = \inf_{Z\in\mathbb{S}^{n+1}} \left\{ \langle M_{\text{obj}}, Z\rangle : \begin{array}{l} Z \in \mathcal{S}(\mathcal{M}) \\ \langle B, Z\rangle = 1 \end{array} \right\}.$$

Related: Hildebrand [2016, Lemma 1.2]

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- $\mathcal{S}(\mathcal{M})$ is ROG $\implies$ objective value exactness.
- $\mathcal{S}(\mathcal{M})$ is ROG $\implies$ closed convex hull exactness via projected SDP set.

Theorem

Given $\mathcal{M} = [m]$, let $\mathcal{X} := \left\{ x \in \mathbb{R}^n : \binom{x}{1}\binom{x}{1}^\top \in \mathcal{S}(\mathcal{M}) \right\}$. and $A(\gamma^*) := \sum_{i\in[m]} \gamma_i^* A_i$.

- If $\mathcal{S}(\mathcal{M})$ is ROG and $\exists \gamma^* \in \mathbb{R}^m_+$ s.t. $A(\gamma^*) \succ 0$, then $\mathrm{conv}(\mathcal{X}) =$ projected SDP domain, i.e.,

$$\mathrm{conv}(\mathcal{X}) = \left\{ x \in \mathbb{R}^n : \begin{array}{l} \exists X, Z \text{ s.t. } Z = \begin{pmatrix} X & x \\ x^\top & 1 \end{pmatrix} \\ Z \in \mathcal{S}(\mathcal{M}) \end{array} \right\}.$$

- If $\mathcal{S}(\mathcal{M})$ is ROG and $\exists \gamma^* \in \mathbb{R}^m_+$ s.t. $A_{\text{obj}} + A(\gamma^*) \succ 0$, then

$$\mathrm{cl\,conv}\left(\left\{ (x,t) \in \mathbb{R}^{n+1} : q_{\text{obj}}(x) \leq t, \ x \in \mathcal{X} \right\}\right) = \mathrm{cl}(\mathcal{D}_{\text{SDP}}).$$

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- Applications when $|\mathcal{M}|$ is not finite
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    [Sturm and Zhang, 2003], [Burer, 2015] and references therein, [Yang et al., 2018]
  - Intersection of two Euclidean balls
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ROG

$$\mathcal{S}(\mathcal{M}) := \left\{ Z \in \mathbb{S}_+^{n+1} : \langle M, Z \rangle \le 0,\ \forall M \in \mathcal{M} \right\}$$

Well-known ROG sets:

- Positive semidefinite cone $\mathbb{S}_+^{n+1}$ itself!

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Theorem (S-lemma)

$\mathcal{S}(\{M\})$ for any $M \in \mathbb{S}^{n+1}$ is ROG.

[Fradkov and Yakubovich, 1979, Sturm and Zhang, 2003]

S-lemma

Corollary (Homogeneous S-lemma)

For any $M_{\text{obj}}, M \in \mathbb{S}^{n+1}$, we have

$$\inf_{z \in \mathbb{R}^{n+1}} \left\{ z^\top M_{\text{obj}} z : z^\top M z \leq 0 \right\} = \inf_{Z \in \mathbb{S}^{n+1}} \left\{ \langle M_{\text{obj}}, Z \rangle : \langle M, Z \rangle \leq 0 \right\}.$$

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Equivalently, suppose $\exists \bar{z}$ s.t. $\bar{z}^\top M \bar{z} < 0$. Then,

$$\left[z^\top M z \leq 0 \implies z^\top M_{\text{obj}} z \leq 0 \right] \text{ iff } \left[\exists \alpha \in \mathbb{R}_+ \text{ s.t. } \alpha M \succeq M_{\text{obj}} \right].$$

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Corollary (Inhomogeneous S-lemma)

For any $A_{\text{obj}}, A \in \mathbb{S}^n$, any $b_{\text{obj}}, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ s.t. $\exists \bar{x}$ satisfying $\bar{x}^\top A \bar{x} + b^\top \bar{x} + c < 0$ and $\text{Opt} > -\infty$, we have

$$\begin{aligned} \text{Opt} &= \inf_{x\in\mathbb{R}^n} \left\{ x^\top A_{\text{obj}} x + b_{\text{obj}}^\top x : x^\top A x + b^\top x + c \leq 0 \right\} \\ &= \inf_{x\in\mathbb{R}^n, X\in\mathbb{S}^n} \left\{ \langle A_{\text{obj}}, X\rangle + b_{\text{obj}}^\top x : \langle A, X\rangle + b^\top x + c \leq 0,\ X \succeq x x^\top \right\}. \end{aligned}$$

When is $\mathcal{S}(\mathcal{M})$ ROG?

- **Question:** for what $\mathcal{M} \subseteq \mathbb{S}^{n+1}$ is $\mathcal{S}(\mathcal{M})$ ROG?

Thank you!

Questions?

References I

Agler, J., Helton, W., McCullough, S., and Rodman, L. (1988). Positive semidefinite matrices with a given sparsity pattern. *Linear Algebra Appl.*, 107:101–149.

Argue, C., K.-K., F., and Wang, A. L. (2022+). Necessary and sufficient conditions for rank-one generated cones. *Math. Oper. Res.*, Forthcoming, (arXiv:2007.07433).

Ben-Tal, A. and Nemirovski, A. (2001). *Lectures on Modern Convex Optimization*, volume 2 of *MPS-SIAM Ser. Optim.* SIAM.

Blekherman, G., Sinn, R., and Velasco, M. (2017). Do sums of squares dream of free resolutions? *SIAM J. Appl. Algebra Geom.*, 1:175–199.

Burer, S. (2015). A gentle, geometric introduction to copositive optimization. *Math. Program.*, 151:89–116.

Burer, S. (2023). A slightly lifted convex relaxation for nonconvex quadratic programming with ball constraints. *arXiv preprint arXiv:2303.01624*.

Fradkov, A. L. and Yakubovich, V. A. (1979). The S-procedure and duality relations in nonconvex problems of quadratic programming. *Vestnik Leningrad Univ. Math.*, 6:101–109.

Grone, R., Johnson, C. R., Sá, E. M., and Wolkowicz, H. (1984). Positive definite completions of partial Hermitian matrices. *Linear Algebra Appl.*, 58:109–124.

References II

Hildebrand, R. (2016). Spectrahedral cones generated by rank 1 matrices. *J. Global Optim.*, 64:349–397.

K.-K., F. and Wang, A. L. (2021). Exactness in SDP relaxations of QCQPs: Theory and applications. Tut. in Oper. Res. INFORMS.

Kelly, S., Ouyang, Y., and Yang, B. (2022). A note on semidefinite representable reformulations for two variants of the trust-region subproblem. *Manuscript, School of Mathematical and Statistical Sciences, Clemson University, Clemson, South Carolina, USA*.

Laurent, M. and Poljak, S. (1995). On a positive semidefinite relaxation of the cut polytope. *Linear Algebra Appl.*, 223-224:439–461.

Paulsen, V. I., Power, S. C., and Smith, R. R. (1989). Schur products and matrix completions. *J. Funct. Anal.*, 85(1):151–178.

Sturm, J. F. and Zhang, S. (2003). On cones of nonnegative quadratic functions. *Math. Oper. Res.*, 28(2):246–267.

Wang, A. L. and K.-K., F. (2022a). Accelerated first-order methods for a class of semidefinite programs. *arXiv preprint*, 2206.00224.

References III

Wang, A. L. and K.-K., F. (2022b). The generalized trust region subproblem: solution complexity and convex hull results. *Math. Program.*, 191:445–486.

Wang, A. L. and K.-K., F. (2022c). On the tightness of SDP relaxations of QCQPs. *Math. Program.*, 193:33–73.

Wang, A. L., Lu, Y., and K.-K., F. (2023+). Implicit regularity and linear convergence rates for the generalized trust-region subproblem. *SIAM J. Optim.*, Forthcoming, (arXiv:2112.13821).

Yang, B., Anstreicher, K., and Burer, S. (2018). Quadratic programs with hollows. *Math. Program.*, 170:541–553.