An Introduction to Semidefinite Program Relaxations of Quadratically Constrained Quadratic Programs

Fatma Kılınç-Karzan

Carnegie Mellon University
Tepper School of Business

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Recap: QCQP and its SDP relaxation

- $$q_i(x) := x^\top A_i x + 2b_i^\top x + c_i = \begin{pmatrix} x \\ 1 \end{pmatrix}^\top \underbrace{\begin{pmatrix} A_i & b_i \\ b_i^\top & c_i \end{pmatrix}}_{=:\, M_i} \begin{pmatrix} x \\ 1 \end{pmatrix} = \left\langle M_i, \begin{pmatrix} xx^\top & x \\ x^\top & 1 \end{pmatrix} \right\rangle$$

- $$\text{Opt} = \inf_{x \in \mathbb{R}^n} \left\{ \left\langle M_{\text{obj}}, \begin{pmatrix} xx^\top & x \\ x^\top & 1 \end{pmatrix} \right\rangle : \left\langle M_i, \begin{pmatrix} xx^\top & x \\ x^\top & 1 \end{pmatrix} \right\rangle \leq 0,\ \forall i \in [m] \right\}$$
$$\geq \inf_{x \in \mathbb{R}^n,\, X \in \mathbb{S}^n,\, Z \in \mathbb{S}^{n+1}} \left\{ \langle M_{\text{obj}}, Z \rangle : \begin{array}{l} \langle M_i, Z \rangle \leq 0,\ \forall i \in [m] \\ Z = \begin{pmatrix} X & x \\ x^\top & 1 \end{pmatrix} \succeq 0 \end{array} \right\} = \text{Opt}_{\text{SDP}}$$

Recap: forms of exactness

- What does exactness mean?
  - Objective value exactness: $\mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}$
    - Optimizer exactness: $\arg\min \mathrm{Opt} = \arg\min \mathrm{Opt}_{\mathsf{SDP}}$
  - Convex hull exactness: $\mathrm{conv}(\mathcal{D}) = \mathcal{D}_{\mathsf{SDP}}$ $\longleftarrow$ convexification of substructures
  - Rank-one generated (ROG) property:

    "SDP exactness that is oblivious to the objective function"

    $\longrightarrow$ exactness in the lifted SDP space

Today's outline

- **Rank-one generated (ROG) property of SDPs**
  - Sufficient (necessary) conditions
  - Examples
- **Exactness in the original space**
  - Convex hull
  - Objective value
- **Efficient algorithms for solving SDPs**

Exactness in the lifted SDP space: ROG property

References:

Argue, C., K.-K., F., and Wang, A. L. (2022+). Necessary and sufficient conditions for rank-one generated cones. *Math. Oper. Res.*, Forthcoming, (arXiv:2007.07433)

K.-K., F. and Wang, A. L. (2021). Exactness in SDP relaxations of QCQPs: Theory and applications. Tut. in Oper. Res. INFORMS

Recap: ROG

- Given $\mathcal{M} \subseteq \mathbb{S}^{n+1}$, define $\mathcal{S}(\mathcal{M}) := \{Z \in \mathbb{S}^{n+1}_{+} : \langle M, Z \rangle \leq 0, \ \forall M \in \mathcal{M}\}$

Definition

A closed cone $\mathcal{S} \subseteq \mathbb{S}^{n+1}_{+}$ is rank-one generated (ROG) if

$$\mathcal{S} = \operatorname{conv}\left(\mathcal{S} \cap \left\{zz^\top : z \in \mathbb{R}^{n+1}\right\}\right).$$

Equivalently, if all extreme rays are generated by rank-one matrices.

- **Analogy:** (Integer programs, integral polyhedra) $\approx$ (QCQPs, ROG)
- ROG implies exactness (objective value and convex hull via the projected SDP)

$$\mathcal{S}(\mathcal{M}) := \{Z \in \mathbb{S}_+^{n+1} : \langle M, Z \rangle \leq 0, \ \forall M \in \mathcal{M}\}$$

Well-known ROG sets:

- Positive semidefinite cone $\mathbb{S}_+^{n+1}$ itself!
- Any single linear matrix inequality (LMI) or equation (LME):

Theorem (S-lemma)

$\mathcal{S}(\{M\})$ for any $M \in \mathbb{S}^{n+1}$ is ROG.

[Fradkov and Yakubovich, 1979, Sturm and Zhang, 2003]

When is $\mathcal{S}(\mathcal{M})$ ROG?

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- Can we analyze ROG property of $\mathcal{S}(\mathcal{M})$ from ROG property of

$$\mathcal{T}(\mathcal{M}) := \left\{ Z \in \mathbb{S}^{n+1}_{+} : \langle M, Z \rangle = 0,\ \forall M \in \mathcal{M} \right\}?$$

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- Caveat:
  - When $\mathcal{M}$ is finite, $\mathcal{S}(\mathcal{M})$ can be "lifted" into $\mathcal{T}(\mathcal{M}')$. But, ROG property is not necessarily preserved in such liftings.

Facial structure

$$\mathcal{S}(\mathcal{M}) = \left\{ Z \in \mathbb{S}^{n+1}_+ : \langle M, Z \rangle \leq 0, \ \forall M \in \mathcal{M} \right\}$$
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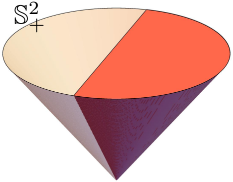
Proposition

- $\boxed{\mathcal{S}(\mathcal{M}) \text{ is ROG}} \iff \boxed{\text{every face of } \mathcal{S}(\mathcal{M}) \text{ is ROG}}$
- $\boxed{\mathcal{S}(\mathcal{M}) \text{ is ROG}} \implies \boxed{\mathcal{T}(\mathcal{M}) \text{ is ROG}}$, $\not\Longleftarrow$
- When $\mathcal{M}$ is compact,
  $\boxed{\mathcal{S}(\mathcal{M}) \text{ is ROG}} \iff \boxed{\forall \varnothing \neq \mathcal{M}' \subseteq \mathcal{M},\ \mathcal{S}(\mathcal{M}) \cap \mathcal{T}(\mathcal{M}') \text{ is ROG}}$
- When $\mathcal{M}$ is finite, $\boxed{\forall \mathcal{M}' \subseteq \mathcal{M},\ \mathcal{T}(\mathcal{M}') \text{ is ROG}} \implies \boxed{\mathcal{S}(\mathcal{M}) \text{ is ROG}}$

Facial structure → sufficient conditions

- $|\mathcal{M}| = 1$, then both $\mathcal{S}(\mathcal{M})$ and $\mathcal{T}(\mathcal{M})$ are ROG (S-lemma)



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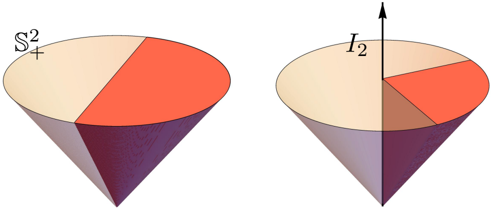

Facial structure $\to$ sufficient conditions

- $|\mathcal{M}| = 1$, then both $\mathcal{S}(\mathcal{M})$ and $\mathcal{T}(\mathcal{M})$ are ROG (S-lemma)
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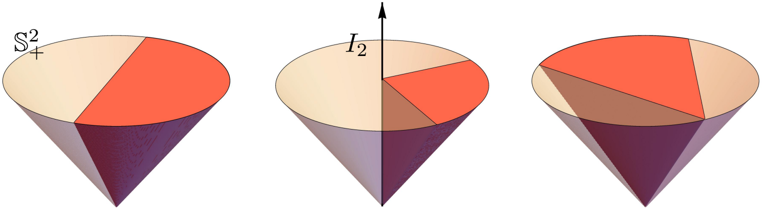
- Two LMIs $\langle M_1, Z\rangle \leq 0$ and $\langle M_2, Z\rangle \leq 0$ are "non-interacting" when

$$\exists(\alpha_1, \alpha_2) \neq (0,0),\ \alpha_1 M_1 + \alpha_2 M_2 \succeq 0$$



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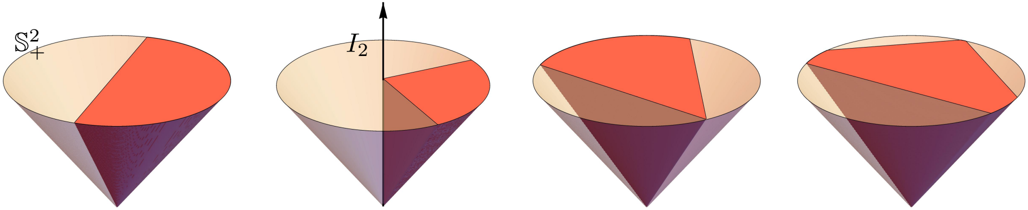
Lemma

If every pair (M_i, M_j) is "non-interacting" in $\mathcal{M} = \{M_1, \ldots, M_k\}$, then $\mathcal{T}(\mathcal{M})$ and $\mathcal{S}(\mathcal{M})$ are ROG.



The ROG property and solutions to quadratic systems

- Let $\mathcal{E}(Z, \mathcal{M}) := \left\{ z \in \mathbb{R}^{n+1} : \langle M, Z \rangle \leq z^\top M z \leq 0, \ \forall M \in \mathcal{M} \right\}$

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- $\mathcal{N}(\mathcal{M}) \subseteq \mathcal{E}(Z,\mathcal{M})$ for all $Z \in \mathcal{S}(\mathcal{M})$
- Suffices to check these for all Z with $\operatorname{rank}(Z) \geq 2$

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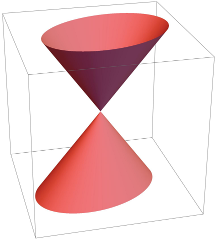
- $\mathcal{N}(\{M\})$ contains $a^\perp$?



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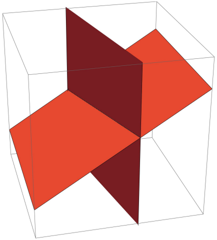

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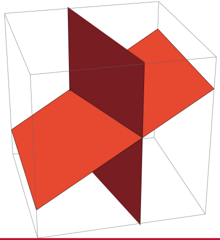

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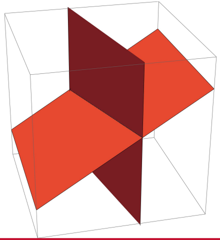

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- For any closed convex cone $\mathbb{K} \subseteq \mathbb{R}^{n+1} \implies \left\{Z \in \mathbb{S}^{n+1}_+ : Za \in \mathbb{K}\right\}$ is ROG.

Summary for two LMIs

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Theorem

$\mathcal{S}(\{M_1, M_2\})$ is ROG if at least one of the following holds

1. $\exists(\alpha_1, \alpha_2) \neq (0,0)$ s.t. $\alpha_1 M_1 + \alpha_2 M_2 \succeq 0$,
2. $\exists a, b_1, b_2 \in \mathbb{R}^{n+1}$ s.t. $M_1 = ab_1^\top + b_1 a^\top$ and $M_2 = ab_2^\top + b_2 a^\top$.

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- Complete characterization of ROG property in the case of two LMIs (c.f., S-lemma).

What other sets are ROG?

- ROG property is extensively studied in the context of **Trust-region subproblem (TRS)** and its variants
  [Sturm and Zhang, 2003], [Burer, 2015] and references therein, [Yang et al., 2018]

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Trust region subproblem

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Trust region subproblem

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$$\ge \inf_{Z\in\mathbb{S}^{n+1}} \left\{ \langle M_{\text{obj}}, Z\rangle : Z_{n+1,n+1} = 1,\ \langle L, Z\rangle \le 0,\ Z \succeq 0 \right\},$$

where $L := \mathrm{Diag}(1, \ldots, 1, -1)$.

Theorem ([Sturm and Zhang, 2003])

$$\mathrm{cl\,conv}\left\{ zz^\top : \left\langle L, zz^\top \right\rangle \le 0 \right\} = \left\{ Z \in \mathbb{S}^{n+1}_+ : \langle L, Z\rangle \le 0 \right\} = \mathcal{S}(\{L\}).$$

(recall $\mathcal{S}(\{L\})$ is ROG).
Furthermore,

$$\mathrm{cl\,conv}\left\{ (x, xx^\top) : \|x\|_2 \le 1 \right\}$$
$$= \left\{ (x, X) : \exists Z = \begin{pmatrix} X & x \\ x^\top & 1 \end{pmatrix},\ Z_{n+1,n+1} = 1,\ \langle L, Z\rangle \le 0,\ Z \succeq 0 \right\}.$$

Extended TRS:

$$\text{(e-TRS):}\ \inf_{x\in\mathbb{R}^n}\left\{ x^\top A_{\mathsf{obj}}x + 2b_{\mathsf{obj}}^\top x :\ \|x\|_2 \le 1,\ c_i^\top \underbrace{\begin{pmatrix} x \\ 1 \end{pmatrix}}_{:= z} \ge 0,\ \forall i \in [m] \right\}$$

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Not really! We need a stronger relaxation.

How to strengthen the standard SDP relaxation?

$$\text{(e-TRS)} = \inf_{z \in \mathbb{R}^{n+1}} \left\{ \left\langle M_{\text{obj}}, zz^\top \right\rangle : \; z_{n+1} = 1, \; \|(z_1, \ldots, z_n)\|_2 \leq 1, \; c_i^\top z \geq 0, \; \forall i \in [m] \right\}$$

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- **Linear RLT:** $\left. \begin{array}{l} c_1^\top z \ge 0 \\ c_2^\top z \ge 0 \end{array} \right\} \implies c_1^\top z z^\top c_2 \ge 0 \implies c_1^\top Z c_2 \ge 0$

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- **SOC RLT:** $\left.\begin{array}{l} c_1^\top z \ge 0 \\ z \in \mathbb{L}^{n+1} \end{array}\right\} \implies zz^\top c_1 \in \mathbb{L}^{n+1} \implies Zc_1 \in \mathbb{L}^{n+1}$

ROG characterization of (e-TRS)

$$\text{(e-TRS)} = \inf_{z\in\mathbb{R}^{n+1}} \left\{ \left\langle M_{\text{obj}}, zz^\top \right\rangle : z_{n+1} = 1,\ z \in \mathbb{L}^{n+1},\ c_i^\top z \geq 0,\ \forall i \in [m] \right\}$$

Theorem ([Burer and Anstreicher, 2013, Burer, 2015] and references therein)

Suppose $c_i^\top z \geq 0$ for $i \in [m]$ are s.t. whenever $\bar{z}$ is feasible to (e-TRS) and $c_\ell^\top \bar{z} = 0$ for some $\ell \in [m]$, then $c_j^\top \bar{z} \geq 0$ for all $j \in [m]$. Then, the set

$$\left\{ Z \in \mathbb{S}_+^{n+1} :\ \langle L, Z\rangle \leq 0,\ Zc_i \in \mathbb{L}^{n+1}, \forall i \in [m],\ c_i^\top Z c_j \geq 0, \forall i,j \in [m] \right\}$$

is ROG and it is equal to $\operatorname{conv}\left\{ zz^\top :\ z \in \mathbb{L}^{n+1},\ c_i^\top z \geq 0, \forall i \in [m] \right\}$.

Intersection of two Euclidean balls

- ROG characterization of the intersection of two Euclidean balls is studied in [Kelly et al., 2022, Burer, 2023]

$$\text{(tb-TRS)} \;=\; \inf_{x\in\mathbb{R}^n} \left\{ x^\top A_{\mathsf{obj}} x + 2b_{\mathsf{obj}}^\top x :\ \|x\|_2 \le 1,\ \|x-c\|_2 \le \tilde{r} \right\}$$

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Theorem ([Burer, 2023], informal)

Consider (tb-TRS) in the (x, t) space. Then, its strengthened $\mathcal{S}(\mathcal{M})$ set which contains 1 LMI from the norm constraint, 2 SOC-RLT constraints, and 1 LME from the linear RLT, is ROG.

What about nonconvex quadratics?

Theorem ([Yang et al., 2018], informal)

Consider the intersection of

- "ball": $\|x\|_2 \leq 1$
- "cuts": $Cx \geq d$
- "holes": $x^\top A_i x + 2b_i^\top x + c_i \geq 0$, where each $A_i \succ 0$, for all $i \in [k]$.

If none of the cuts and holes touch each other, then the strengthened $\mathcal{S}(\mathcal{M})$ set which contains

- 1 LMI from the norm constraint,
- all SOC-RLT and linear RLT constraints from the cuts, and
- all LMIs $\langle A_i, X \rangle + 2b_i^\top x + c_i \geq 0$ from the holes,

is ROG.

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Given $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^m$, what is the set $\mathcal{S}(\mathcal{M})$ that gives the ROG characterization of

$$\{x \in \mathbb{R}^n : \|x\|_2 \leq 1,\ \|Ax - c\|_2 \leq \tilde{r}\}?$$

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- What about cuts? holes?
- Fejes-Tóth conjecture (1964) (one of Kurt Anstreicher's favorite open problems that can significantly simplify the proof of Kepler conjecture)

Lemma

Suppose

- $\mathcal{M} = \bigcup_{\alpha \in \mathcal{F}} \mathcal{M}_\alpha$ for some family of matrices $\{\mathcal{M}_\alpha\}_{\alpha \in \mathcal{F}}$, and
- $\mathcal{S}(\mathcal{M}_\alpha)$ is ROG for every $\alpha \in \mathcal{F}$.

Then, $\mathcal{S}(\mathcal{M})$ is ROG iff $\operatorname{extr}(\mathcal{S}(\mathcal{M})) \subseteq \bigcap_{\alpha \in \mathcal{F}} \operatorname{extr}(\mathcal{S}(\mathcal{M}_\alpha))$.

Simple ROG preserving operations

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Lemma

Suppose

- $\mathcal{M} = \bigcup_{i=1}^{k} \mathcal{M}_i$, i.e., a *finite* union of compact sets, and
- the following "non-interacting" assumption holds:

  for all $0 \neq Z \in \mathbb{S}^{n+1}_{+}$ *and* $i \in [k]$, *if* $\langle M_i, Z\rangle = 0$ *for some* $M_i \in \mathcal{M}_i$, *then* $\langle M, Z\rangle < 0$ *for all* $M \in \mathcal{M} \setminus \mathcal{M}_i$.

Then, $\mathcal{S}(\mathcal{M})$ is ROG iff $\mathcal{S}(\mathcal{M}_i)$ is ROG for all $i \in [k]$.

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- ROG property implies (closed) convex hull exactness for any* objective function

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- Many ROG sets arising from variants of TRS. . .
- Many more open questions about ROG characterizations of sets defined by quadratics. . .

Exactness in the original space

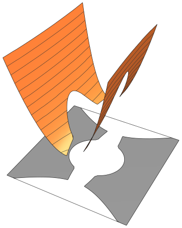
References:

Wang, A. L. and K.-K., F. (2022c). On the tightness of SDP relaxations of QCQPs. *Math. Program.*, 193:33–73

Wang, A. L. and K.-K., F. (2020). A geometric view of SDP exactness in QCQPs and its applications. *arXiv preprint*, 2011.07155

The QCQP epigraph

- QCQP epigraph $\mathcal{D} := \left\{ (x,t) \in \mathbb{R}^{n+1} : \begin{array}{l} q_{\text{obj}}(x) \le t \\ q_i(x) \le 0,\ \forall i \in [m] \end{array} \right\}$



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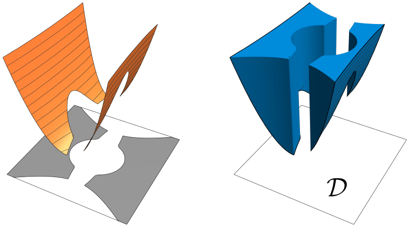

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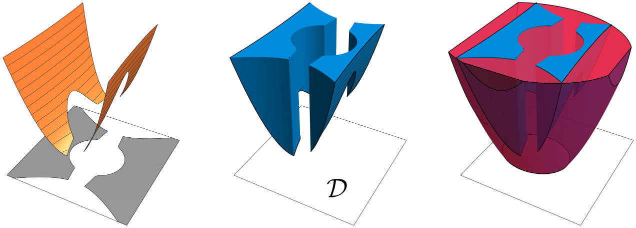

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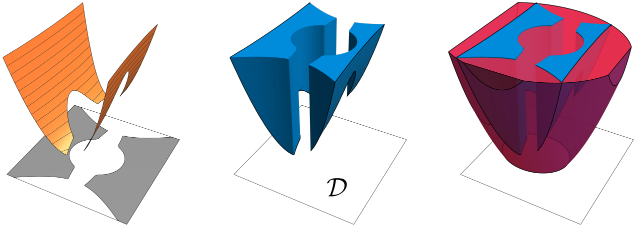

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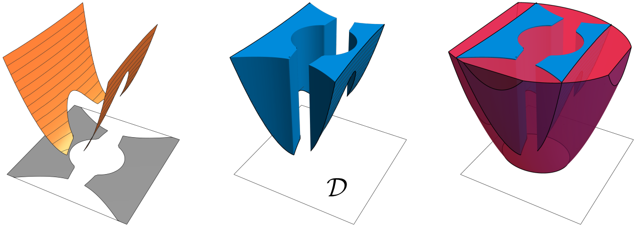

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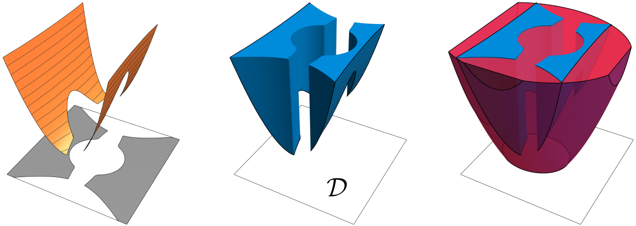

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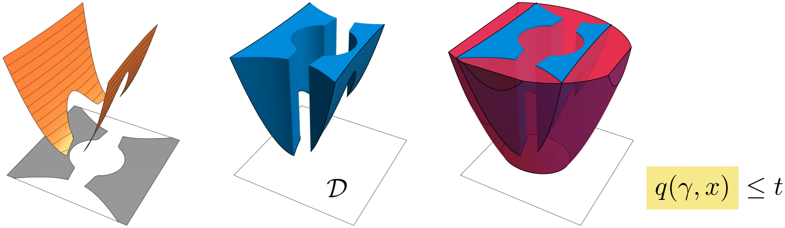

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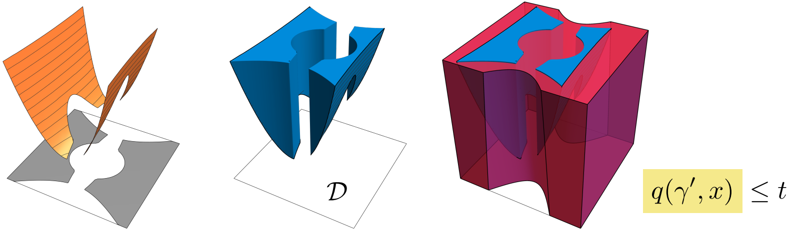

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Dual strict feasibility holds, i.e., $\exists \gamma^* \in \mathbb{R}^m_+$ s.t. $A_{\text{obj}} + \sum_{i \in [m]} \gamma_i^* A_i \succ 0$.

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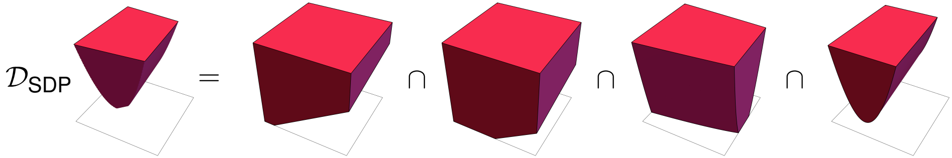

Rewriting SDP in terms of Γ

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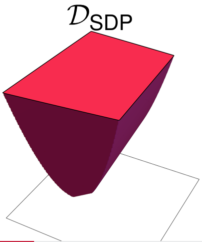

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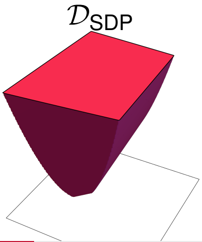

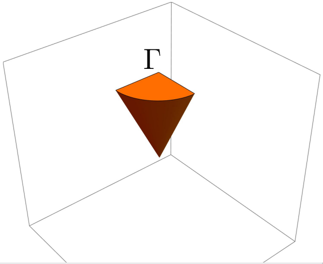

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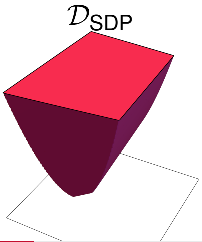

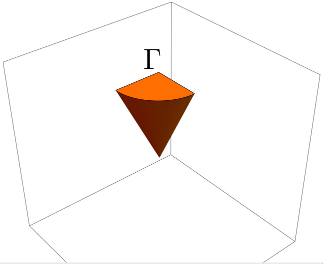

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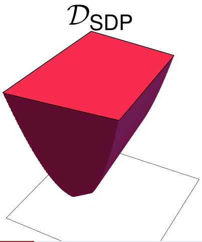

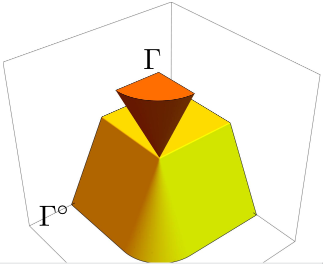

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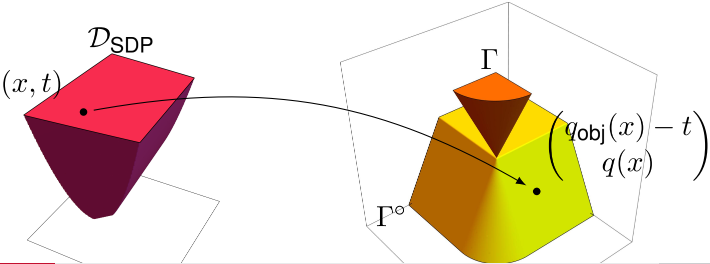

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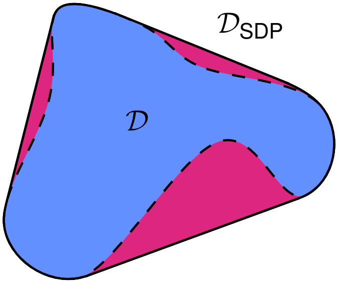

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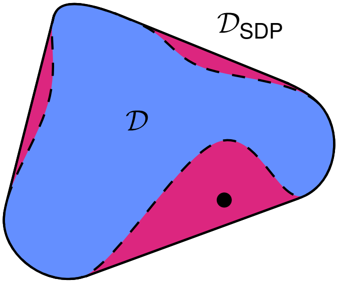

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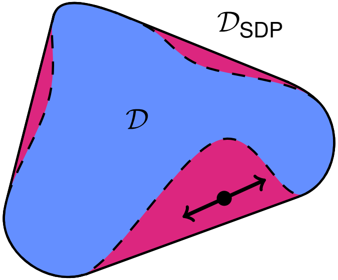

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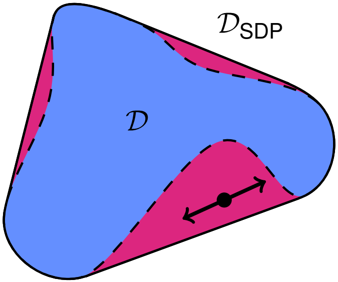

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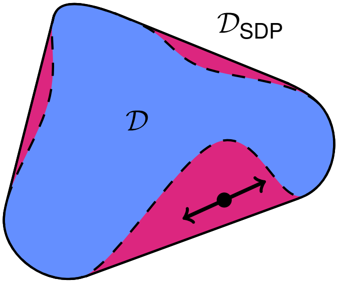

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- When do these "rounding" directions exist? $\longleftarrow$ Can carry out this idea for QCQPs!

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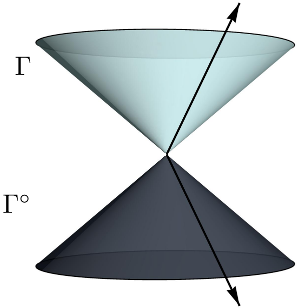

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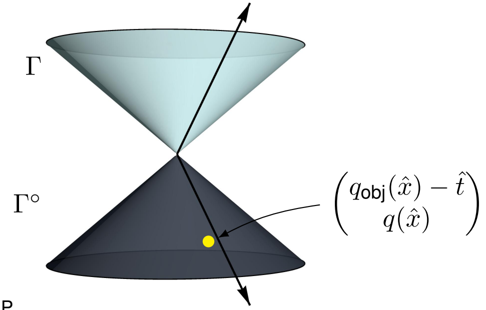

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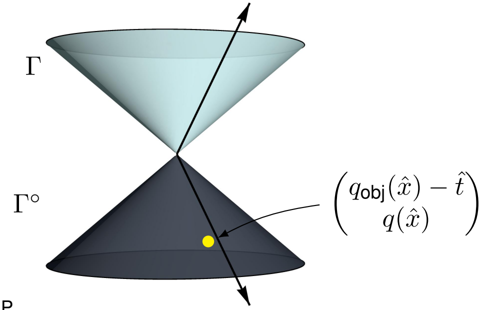

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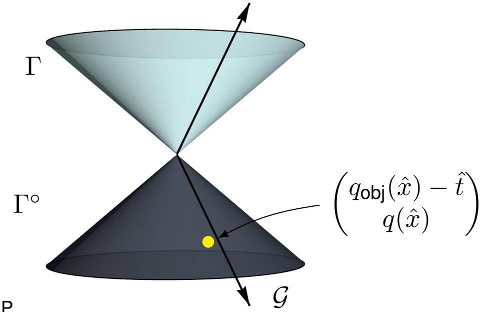

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Faces of Γ and Γ°

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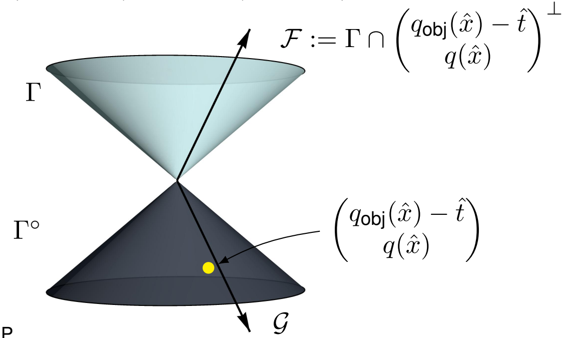

- $\text{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}}$

  $\Longleftrightarrow \forall (\hat{x}, \hat{t}) \in \mathcal{D}_{\text{SDP}} \setminus \mathcal{D}, \exists (x', t') \neq 0$ s.t. $\begin{pmatrix} q_{\text{obj}}(\hat{x} \pm \alpha x') - (\hat{t} \pm \alpha t') \\ q(\hat{x} \pm \alpha x') \end{pmatrix} \in \Gamma^\circ$ for some $\alpha > 0$.

- Let $\mathcal{G}(\hat{x}, \hat{t})$ denote the minimal face of Γ° containing $\begin{pmatrix} q_{\text{obj}}(\hat{x}) - \hat{t} \\ q(\hat{x}) \end{pmatrix}$.

Sufficient condition for convex hull exactness

- Given $(\hat{x}, \hat{t}) \in \mathcal{D}_{\text{SDP}}$, look for a subset of directions (x', t') s.t. $\left[(\hat{x}, \hat{t}) \pm \alpha(x', t')\right] \in \mathcal{D}_{\text{SDP}}$ for some $\alpha > 0$

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Theorem

If for every $(\hat{x}, \hat{t}) \in \mathcal{D}_{\text{SDP}} \setminus \mathcal{D}$, the set

$$\mathcal{R}'(\hat{x}, \hat{t}) := \left\{ (x', t') \in \mathbb{R}^{n+1} : \begin{pmatrix} q_{\text{obj}}(\hat{x} + \alpha x') - (\hat{t} + \alpha t') \\ q(\hat{x} + \alpha x') \end{pmatrix} \in \operatorname{span}(\mathcal{G}(\hat{x}, \hat{t})), \forall \alpha \in \mathbb{R} \right\}.$$

is nontrivial, then $\operatorname{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}}$.

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- When Γ° is facially exposed, $\mathcal{R}'(x,t)$ admits further simplification.
- For example,

Proposition

Suppose Γ is strictly feasible. Consider any $(x,t) \in \mathcal{D}_{\text{SDP}}$ with $t = \sup_{\gamma \in \Gamma_1} q(\gamma, x)$, and let $(1,f) \in \operatorname{rint}(\mathcal{F}(x,t))$.
If Γ is polyhedral, then

$$\mathcal{R}'(x,t) = \left\{ (x',t') \in \mathbb{R}^{n+1} : \begin{array}{l} x' \in \ker(A(f)), \\ \langle b(\gamma), x' \rangle - t' = 0, \ \forall (1,\gamma) \in \mathcal{F}(x,t) \end{array} \right\}.$$

Example: SDP convex hull exactness for $m = 2$

- Consider $\mathcal{X} = \{x : q_i(x) \leq 0, \forall i \in [2]\}$, i.e., $q_{\text{obj}} = 0$ and $m = 2$.

Related: Yıldıran [2009]

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- Suppose $\exists \gamma^* \in \mathbb{R}^2_+$ s.t. $\gamma_1^* A_1 + \gamma_2^* A_2 \succ 0$, and let $\gamma^{(1)}, \gamma^{(2)} \in \mathbb{R}^2_+$ be generators of Γ_1.

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- Then, $\operatorname{cl}\operatorname{conv}(\mathcal{X}) = \mathcal{X}_{\text{SDP}}$ if and only if for both $i = 1, 2$, we have that

$$\ker(A(\gamma^{(i)})) \cap b(\gamma^{(i)})^\perp \text{ is nontrivial.}$$

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Example: QCQPs with symmetry

- Convex hull exactness in the case of "highly symmetric" QCQPs, a.k.a., **quadratic matrix programs** (QMPs):

$$x \in \mathbb{R}^n \quad \longrightarrow \quad X \in \mathbb{R}^{n\times k} \text{ and}$$

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- **Applications**:
  - Robust least squares, sphere packing problems, QCQPs with spherical constraints, orthogonal Procrustes problem

Related: Beck [2007], Beck et al. [2012]

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- Can be written as a QCQP by defining $A_{\text{obj}} = I_k \otimes \mathbb{A}_{\text{obj}}$, $A_i = I_k \otimes \mathbb{A}_i$ $\forall i \in [m]$

$$A = I_k \otimes \mathbb{A} = \begin{pmatrix} \mathbb{A} & & \\ & \ddots & \\ & & \mathbb{A} \end{pmatrix}$$

- Convex hull exactness holds whenever $k \ge m$

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Sufficient condition for objective value exactness

Objective value exactness has been studied a lot:

- TRS and S-lemma

  [Yakubovich, 1971]

- Extended TRS

  [Jeyakumar and Li, 2014, Ben-Tal and den Hertog, 2014, Locatelli, 2016, Ho-Nguyen and K.-K., 2017, Bomze et al., 2018]

- Sign-definite SDPs

  [Sojoudi and Lavaei, 2014]

- SDPs with simultaneously diagonalizable matrices

  [Burer and Ye, 2019, Locatelli, 2022]

- SDPs with certain sparsity patterns (forest, bipartite)

  [Azuma et al., 2022b,a]

- ...

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Theorem

Suppose $\sup_{\gamma\in\Gamma_1} \mathbf{d}(\gamma)$ is achieved at some γ^* for which $A_{\text{obj}} + A(\gamma^*) \succ 0$. Then, $\arg\min_{(x,t)\in\mathcal{D}} t = \arg\min_{(x,t)\in\mathcal{D}_{\text{SDP}}} t$.

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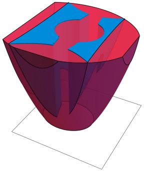
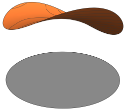
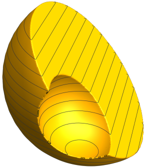
SDPs provide exact reformulations for broad classes of QCQPs!

Efficient algorithms for exact SDPs

References:

Wang, A. L., Lu, Y., and K.-K., F. (2023+). Implicit regularity and linear convergence rates for the generalized trust-region subproblem. *SIAM J. Optim.*, Forthcoming, (arXiv:2112.13821)

Wang, A. L. and K.-K., F. (2022a). Accelerated first-order methods for a class of semidefinite programs. *arXiv preprint*, 2206.00224

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  - Recent theory showing that under some regularity conditions, for almost all objective functions, B-M method finds the global optimum. [Boumal et al., 2016, 2020]

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This is a minimization problem in the original space $\longrightarrow$ n variables
- Exactness (regularity) will allow us to efficiently deal with max-type obj. structure

Linear-time algorithm for the Generalized TRS

- Generalized TRS: $$\mathrm{Opt} = \inf_{x\in\mathbb{R}^n} \left\{ q_{\mathsf{obj}}(x) : q_1(x) \le 0 \right\}$$

Related: Hazan and Koren [2016], Ho-Nguyen and K.-K. [2017], Jiang and Li [2019, 2020]

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- where $\Gamma_1 = \{\gamma \in \mathbb{R}_+ : q(\gamma, x) \text{ is convex in } x\}$

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- Generalized TRS: $\text{Opt} = \inf_{x \in \mathbb{R}^n} \{q_{\text{obj}}(x) : q_1(x) \le 0\}$
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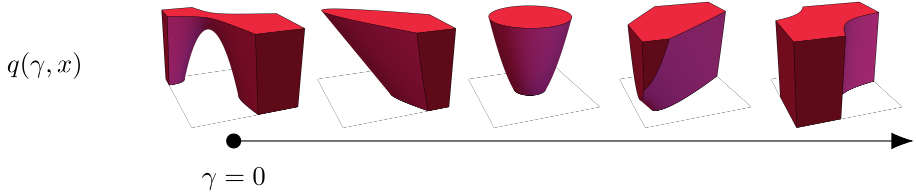

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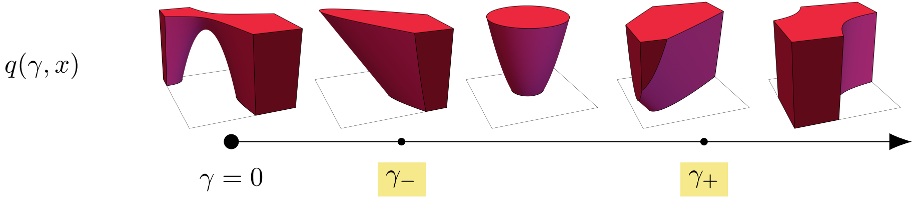

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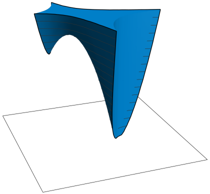

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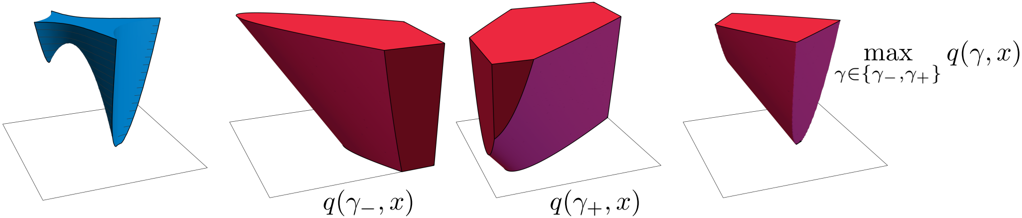

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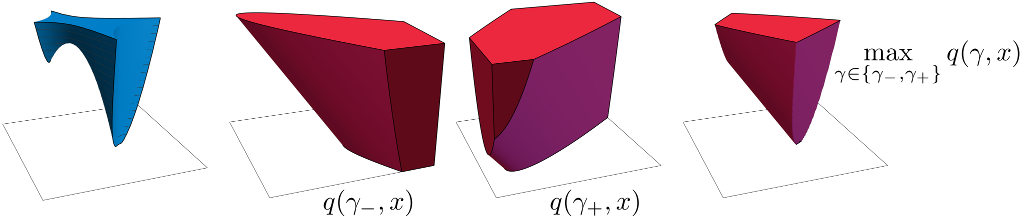

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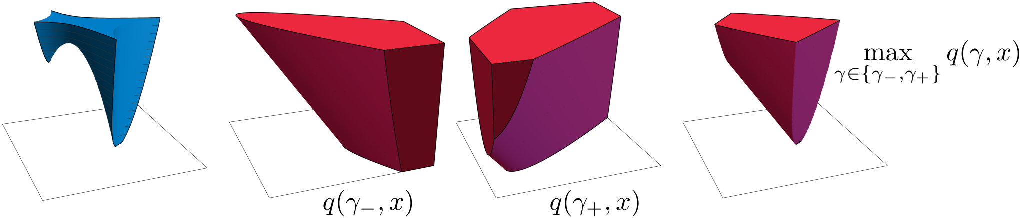

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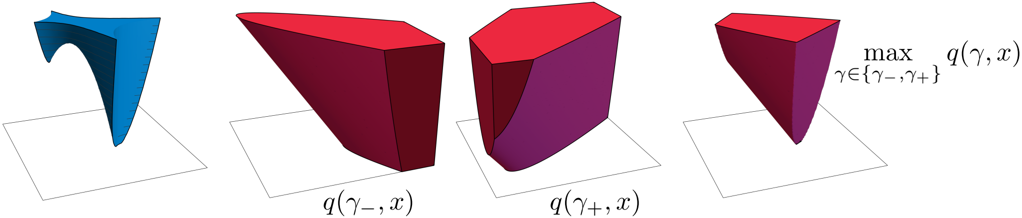

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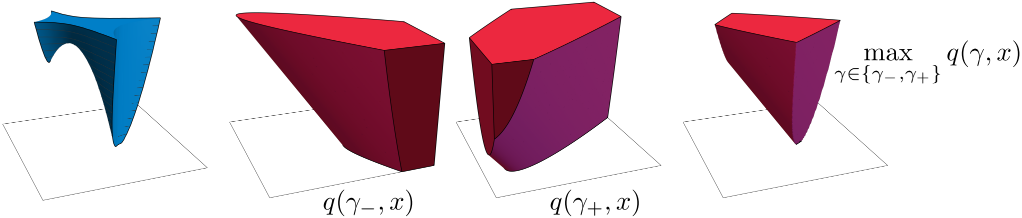

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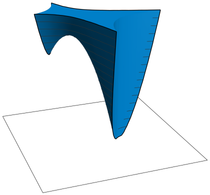

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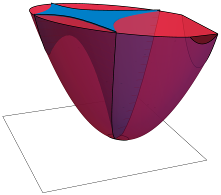

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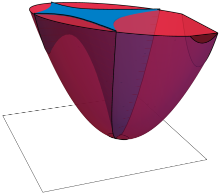

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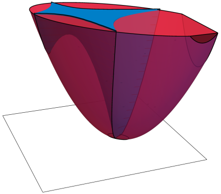

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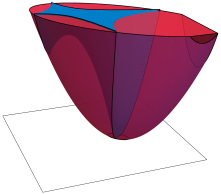

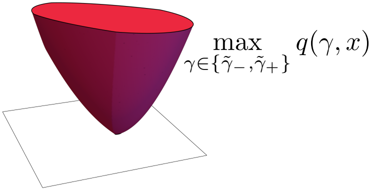

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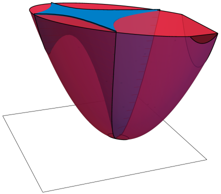

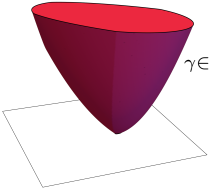

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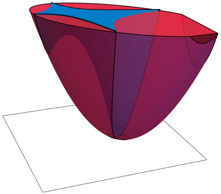

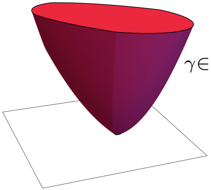

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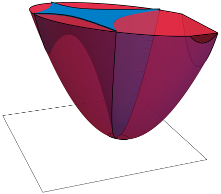

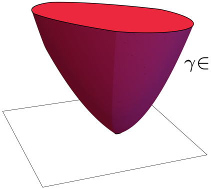

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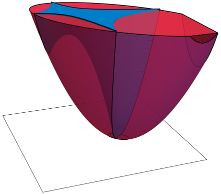

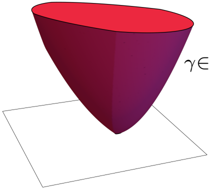

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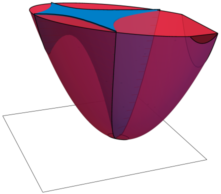

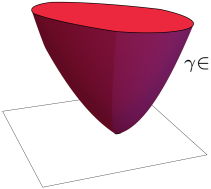

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- For QCQPs, we desire rank-1 solutions in the SDP relaxations.
  What about SDPs in which we seek rank-k solutions?

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- SDP relaxation of QMP in $X \in \mathbb{R}^{(n-k)\times k}$

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- SDP relaxation of QMP in $X \in \mathbb{R}^{(n-k)\times k}$, $\begin{pmatrix} XX^\top & X \\ X^\top & I_k \end{pmatrix} \mapsto \begin{pmatrix} * & * \\ * & I_k \end{pmatrix}$

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- Equivalently, k-exact SDPs originate from QCQPs and QMPs that admit exact SDP relaxations

Deriving a strongly convex minimax problem

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Algorithms

- Given $\mathcal{U}$, how to solve strongly convex QMMP

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Related: Nesterov [2005], Devolder et al. [2013, 2014], Nesterov [2018]

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$\longrightarrow$ CautiousAGD: $O\left(\epsilon^{-1/2}\log(\epsilon^{-1})\right)$

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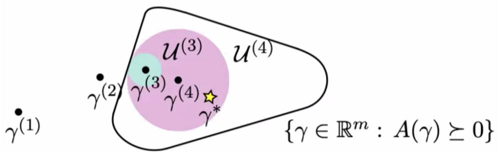

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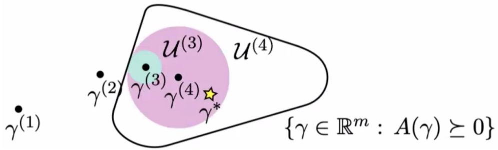
  - Generate $\gamma^{(i)} \to \gamma^*$ and neighborhoods $\mathcal{U}^{(i)} \subseteq \{\gamma : A(\gamma) \succeq 0\}$ and monitor convergence of CautiousAGD for QMMP with $\mathcal{U}^{(i)}$.

  $\longrightarrow$ CertSDP



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CertSDP guarantees

Theorem

Given $\epsilon > 0$, CertSDP produces iterates X_t such that

$$\left\langle M_{\text{obj}}, \begin{pmatrix} X_t X_t^\top & X_t \\ X_t^\top & I_k \end{pmatrix} \right\rangle \leq \mathrm{Opt}_{\text{SDP}} + \epsilon \qquad \text{and} \qquad \left\| \left(\left\langle M_i, \begin{pmatrix} X_t X_t^\top & X_t \\ X_t^\top & I_k \end{pmatrix} \right\rangle + d_i \right)_i \right\|_2 \leq \epsilon.$$

after completing

- **iteration count**: $t \approx O(1) + O\left(\log(\epsilon^{-1})\right)$

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- **storage**: $O(m + nk)$ entries

A glimpse on numerical results

- Random instances of k-exact **distance-minimization QMP**

$$\inf_{X \in \mathbb{R}^{(n-k)\times k}} \left\{ \|X\|_F^2 : \ q_i(X) = 0, \ \forall i \in [m] \right\}$$

with $k = m = 10$, $(n-k) = 10^3, 10^4, 10^5$ (10 instances per setting)

Related: Ding et al. [2021], Yurtsever et al. [2021], Souto et al. [2020], O'Donoghue et al. [2016]

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Algorithm	time (s)	std.	$\|X - X^*\|_F^2$	std.	memory (MB)	std.
CertSDP	1.3×10^3	7.6×10^2	1.9×10^{-22}	4.2×10^{-23}	0.0	0.0
CSSDP	3.0×10^3	5.8×10^{-1}	7.3×10^{-2}	3.4×10^{-2}	0.0	0.0
SketchyCGAL	3.0×10^3	8.5	1.1	6.6×10^{-1}	1.0×10^1	1.0×10^1
ProxSDP	2.1×10^2	1.1×10^1	1.2×10^{-19}	3.2×10^{-19}	4.8×10^1	1.9×10^1
SCS	3.1×10^3	2.5×10^1	5.1×10^{-5}	9.5×10^{-5}	5.3×10^2	4.3×10^1

$n - k = 10^3$, time limit 3×10^3 seconds

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Algorithm	time (s)	std.	$\|X - X^*\|_F^2$	std.	memory (MB)	std.
CertSDP	4.5×10^3	7.0×10^2	1.9×10^{-22}	5.2×10^{-23}	8.5	1.2×10^1
CSSDP	1.0×10^4	6.6×10^{-1}	2.7	9.4×10^{-1}	6.2	1.5×10^1
SketchyCGAL	9.7×10^3	1.8×10^2	4.0	1.4	2.7×10^1	2.2×10^1
ProxSDP	1.2×10^4	1.1×10^2	2.9	9.9×10^{-1}	1.9×10^4	1.2×10^2

$n - k = 10^4$, time limit 10^4 seconds

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Algorithm	time (s)	std.	$\|X - X^*\|_F^2$	std.	memory (MB)	std.
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CSSDP	5.0×10^4	4.7	2.8	5.1×10^{-1}	2.0×10^2	2.5×10^2
SketchyCGAL	4.7×10^4	3.3×10^3	4.0	2.1	3.7×10^2	2.0×10^2

$n - k = 10^5$, time limit 5×10^4 seconds

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- **Future directions**:
  - Can we approach **approximation quality** similarly?

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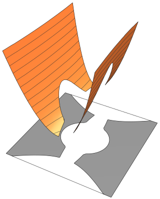
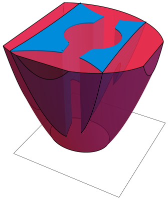
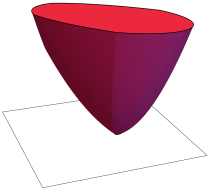
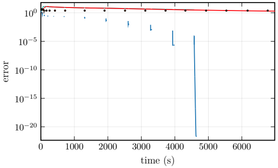
Thank you!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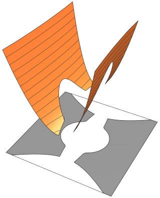
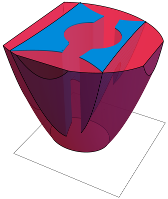
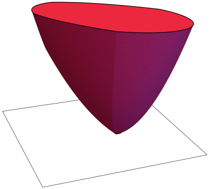
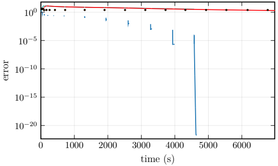
Questions?









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