Constant-Competitive Random Assignment MSP Without Knowing the Matroid

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[1] ETH Zürich, Switzerland

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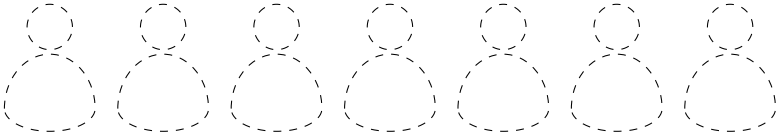
Classical Secretary Problem



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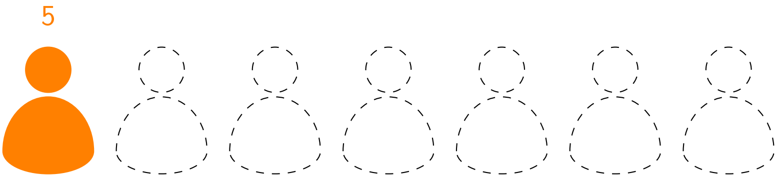

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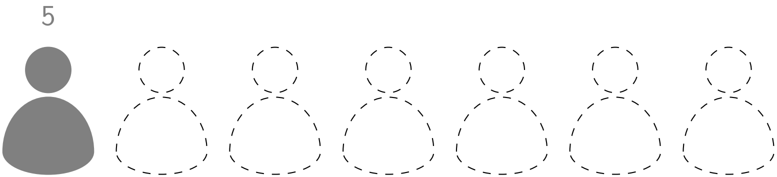

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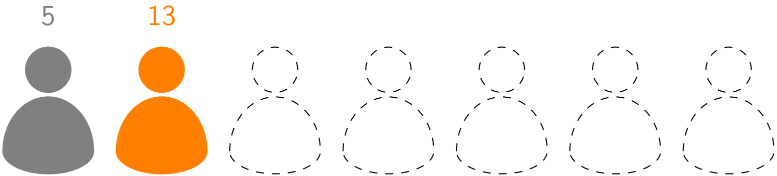

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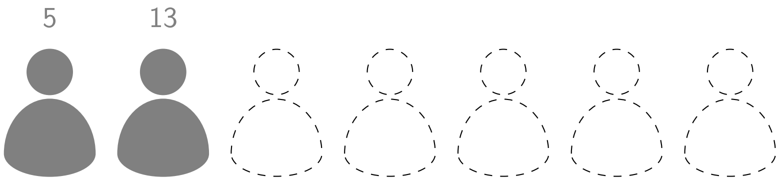

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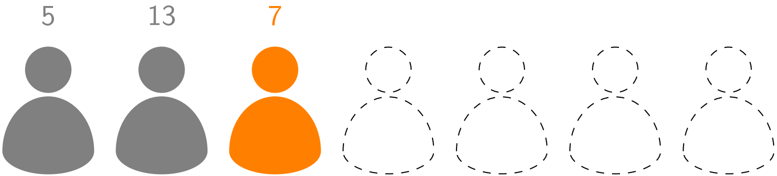

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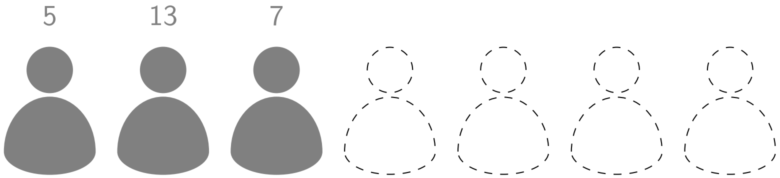

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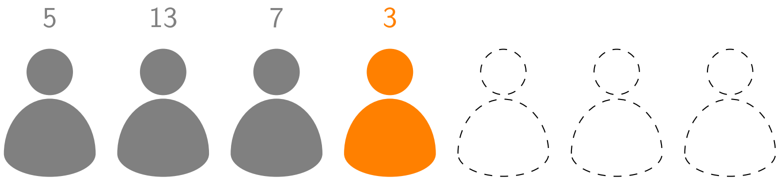

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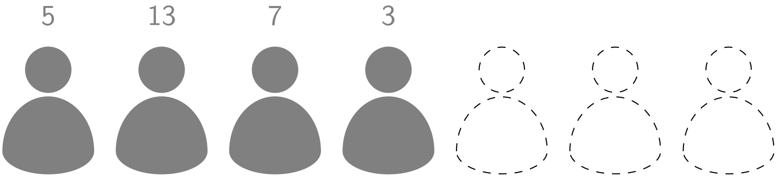

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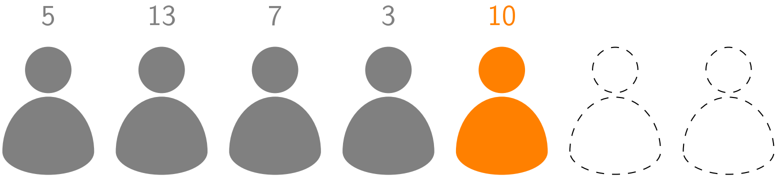

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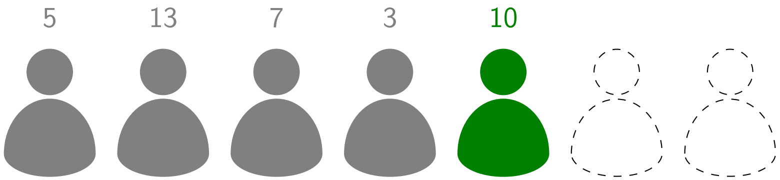

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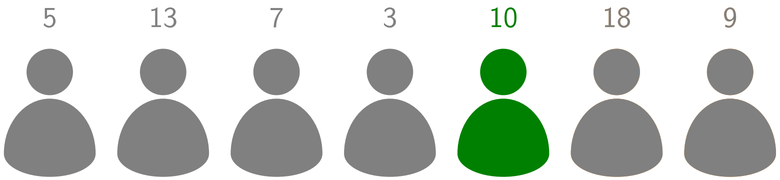

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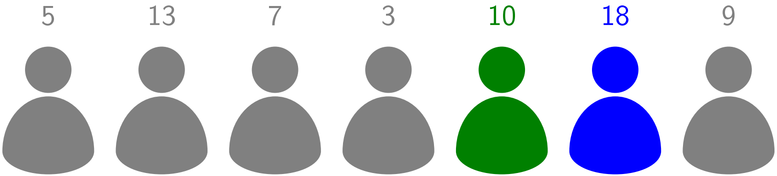

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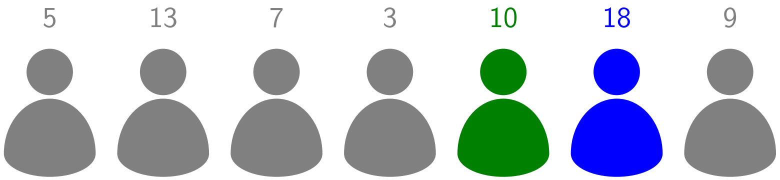

$$\text{Competitive Ratio} = \mathbb{E}\left[\text{selected}\right] / \text{optimal}$$

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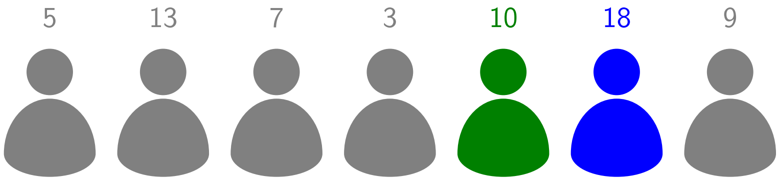

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Theorem [Dynkin 1963]

There exists an optimal $\frac{1}{e}$-competitive algorithm.

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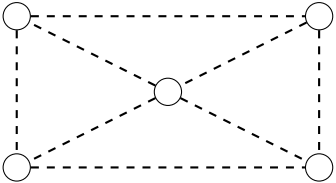

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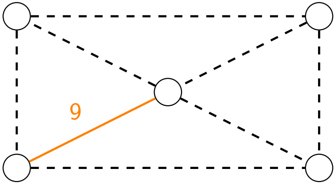

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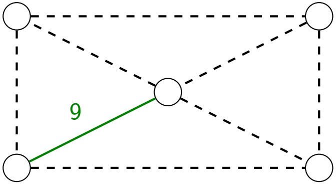

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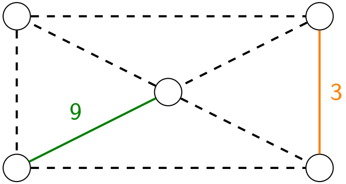

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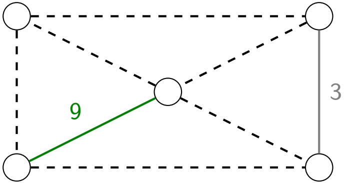

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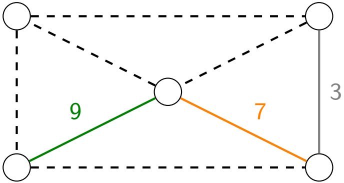

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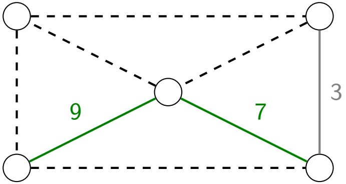

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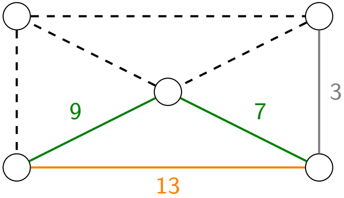

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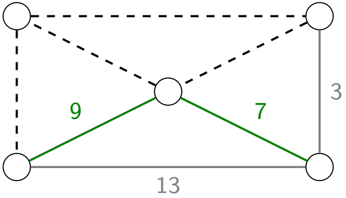

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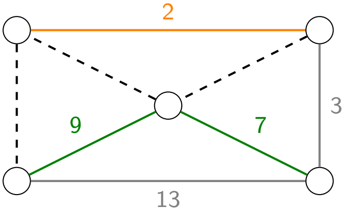

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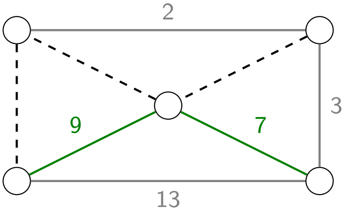

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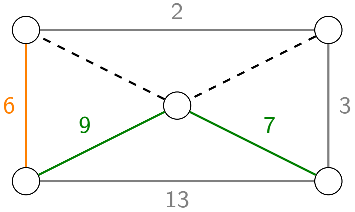

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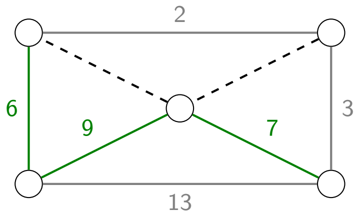

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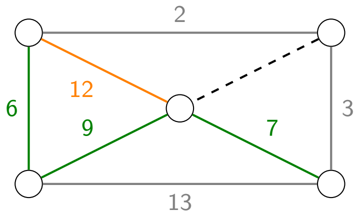

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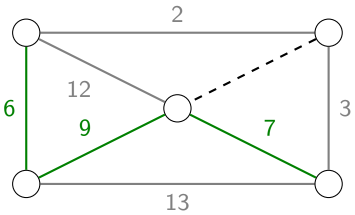

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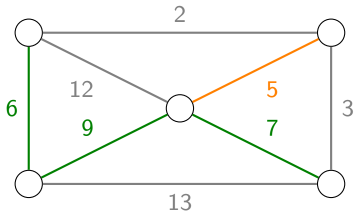

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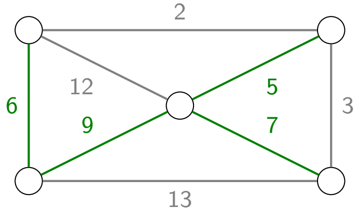

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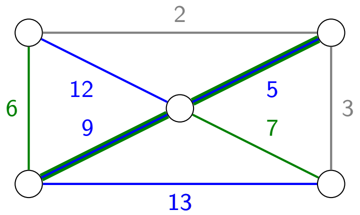

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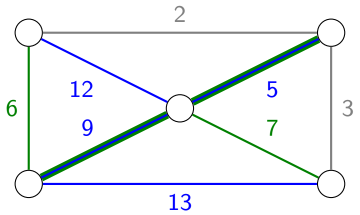

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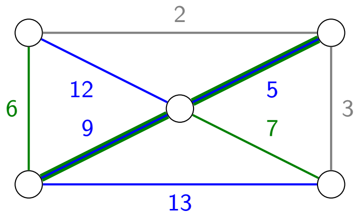

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Conjecture [Babaioff, Immorlica, Kleinberg 2007]

There exists a constant-competitive algorithm for general matroids.

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  - [Lachish 2014], [Feldman, Svensson, Zenklusen 2014]

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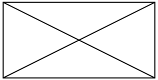

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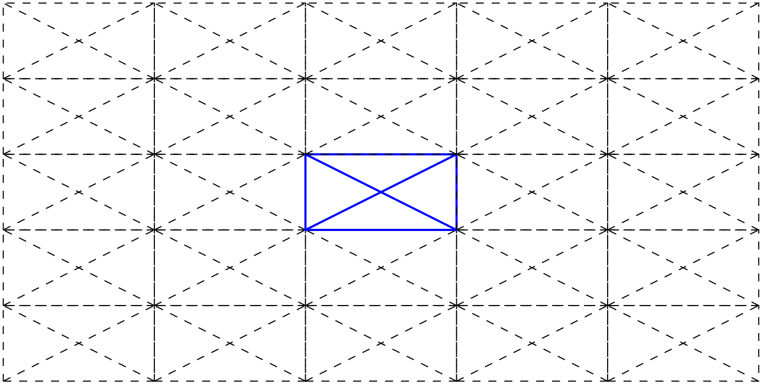

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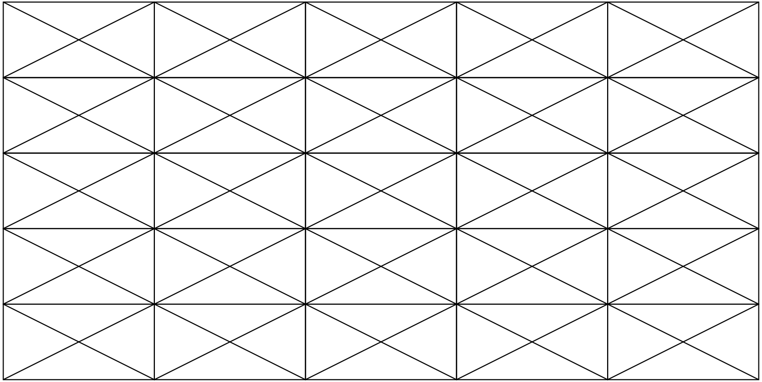

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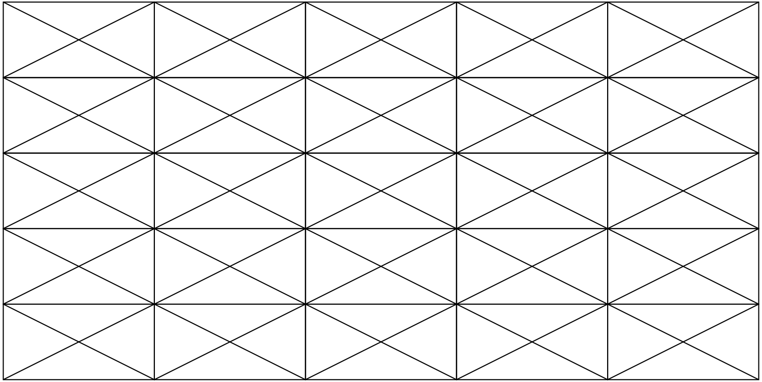

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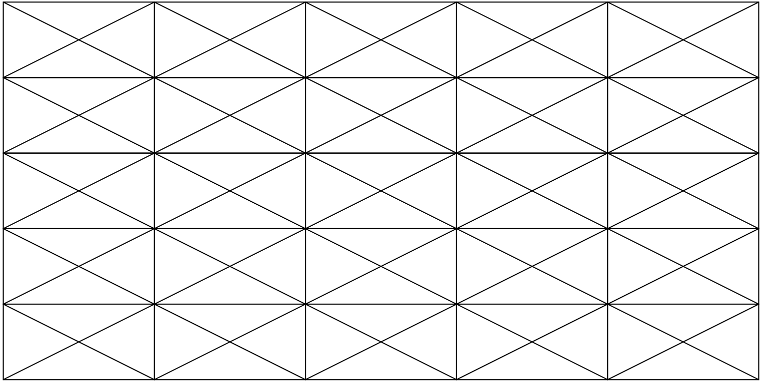

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Goal: Reduce amount of required prior information in RAMSP

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- **Our main result:**
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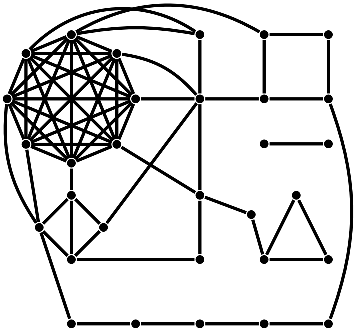
Approach from [Soto 2013]



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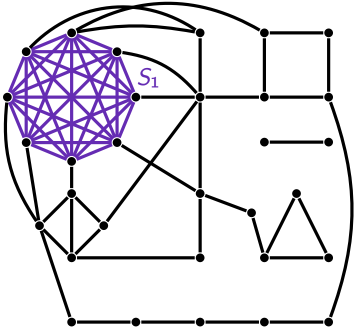

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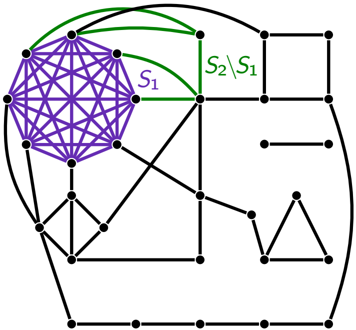

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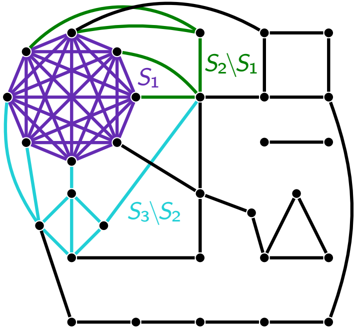

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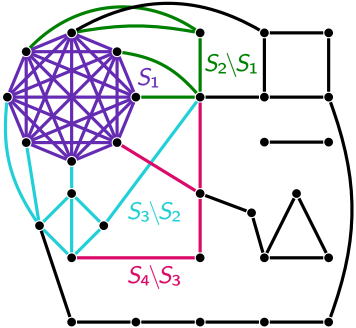

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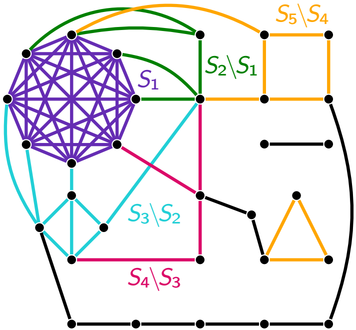

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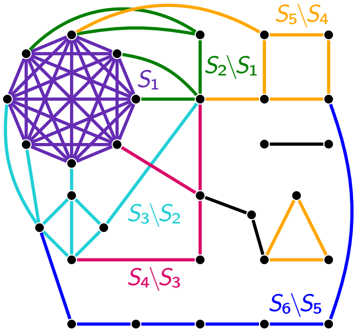

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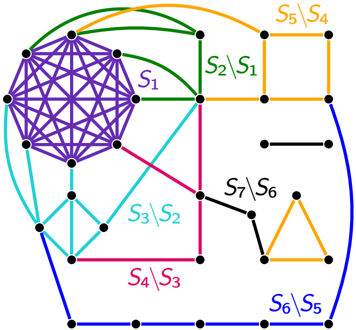

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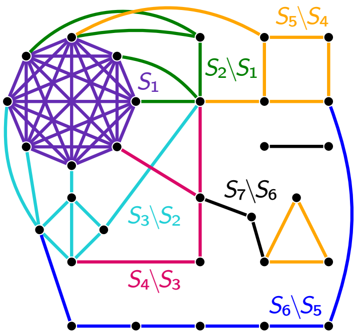

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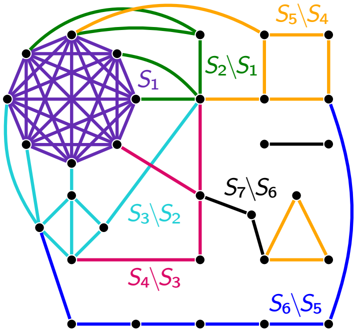

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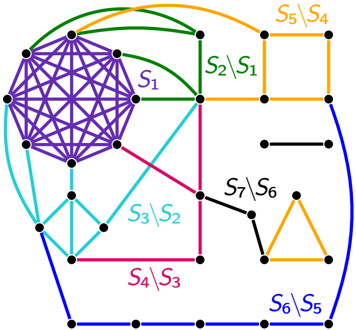

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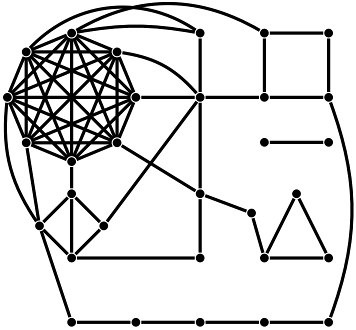

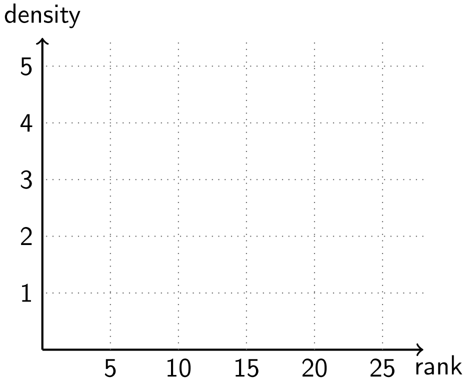

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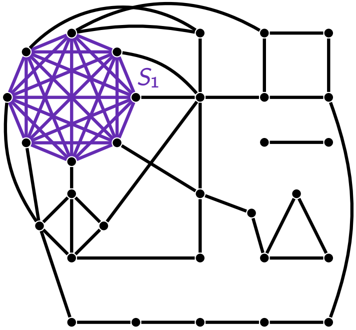

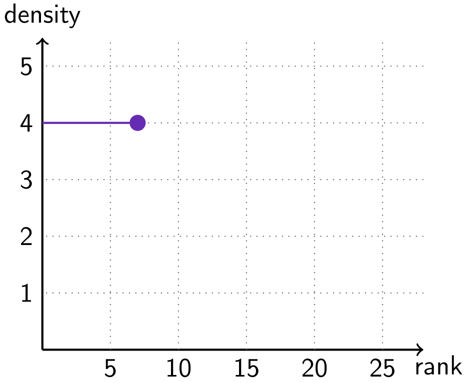

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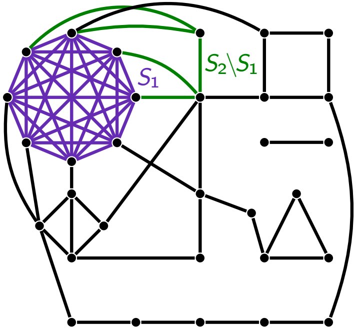

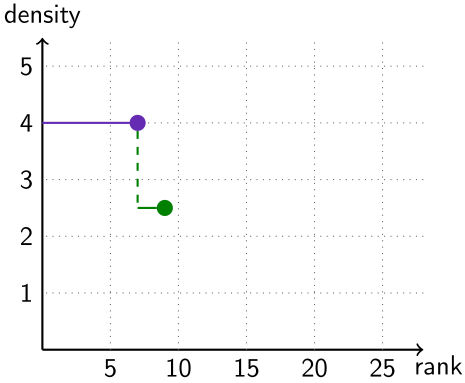

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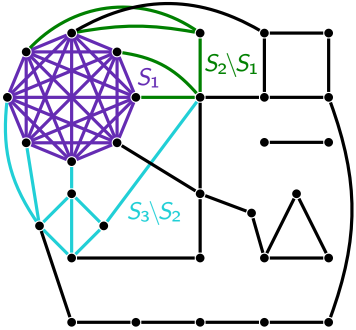

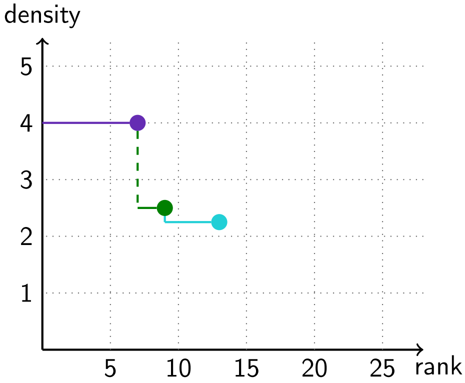

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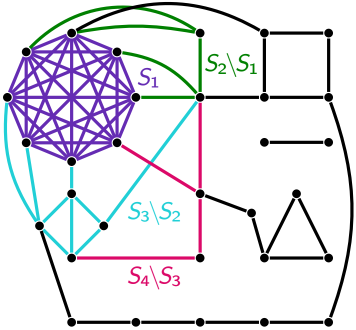

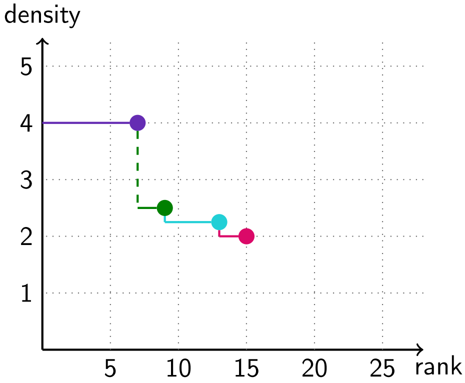

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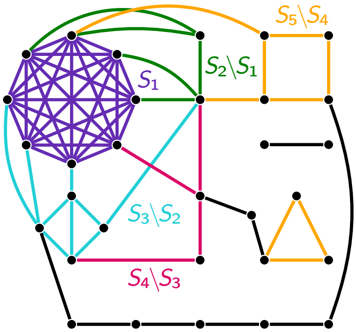

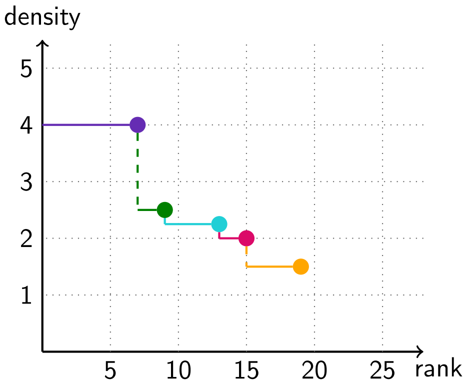

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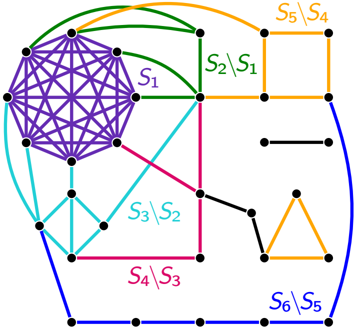

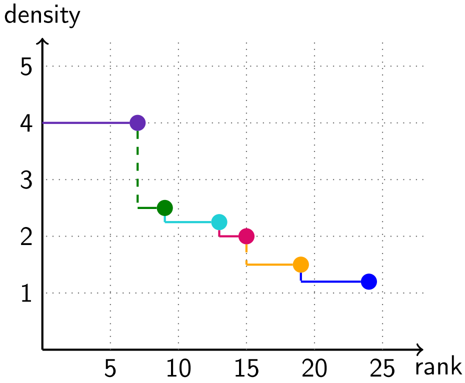

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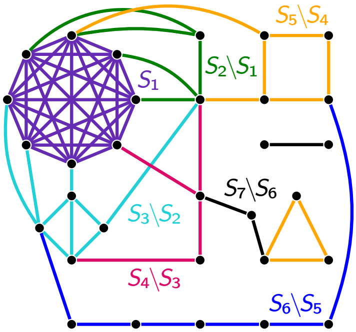

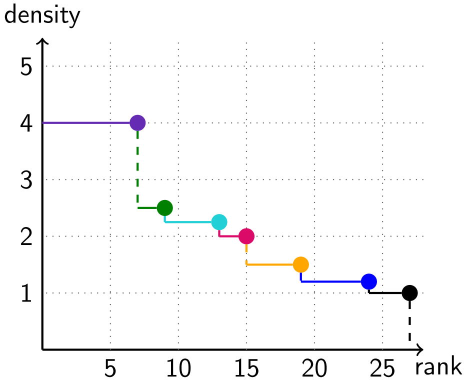

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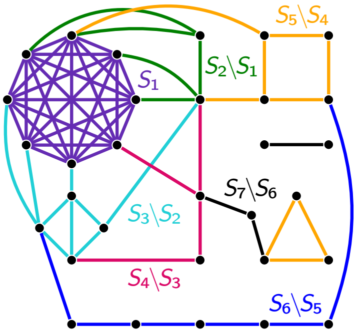

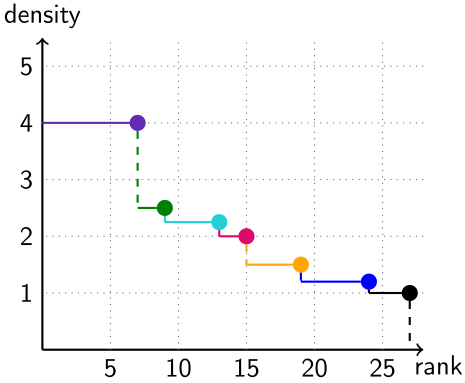

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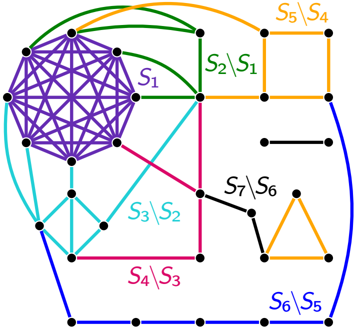

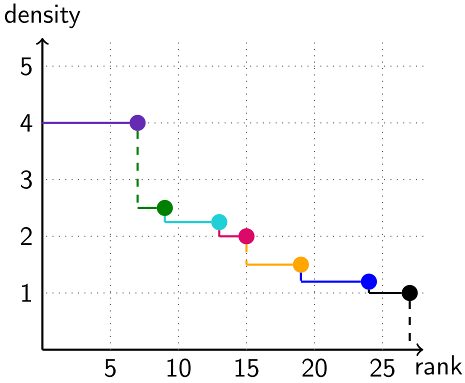

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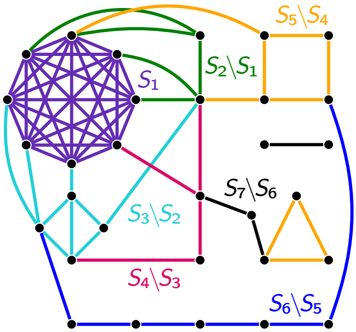

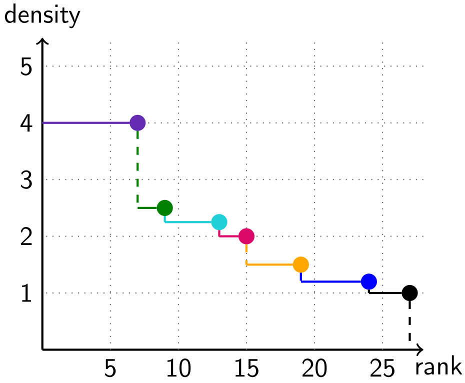

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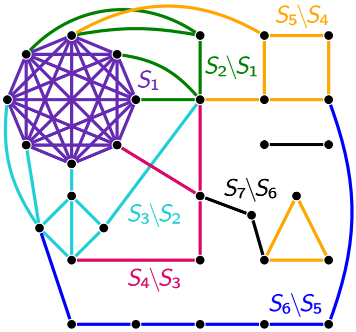

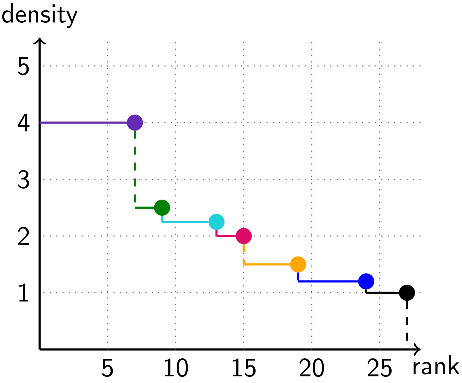

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