Towards a characterization of maximal quadratic-free sets

Gonzalo Muñoz, Joseph Paat and Felipe Serrano

June 23, 2023 - IPCO 2023

How to get a non-linear continuous talk at IPCO

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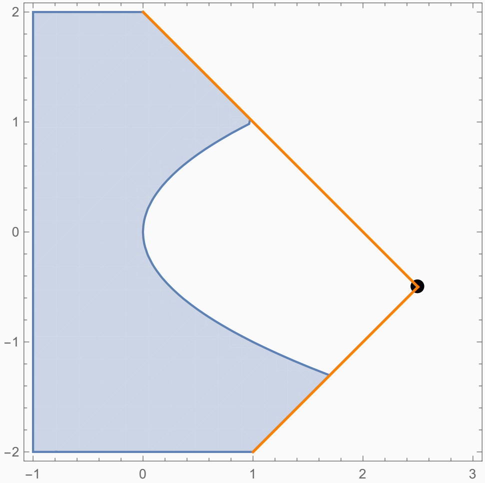
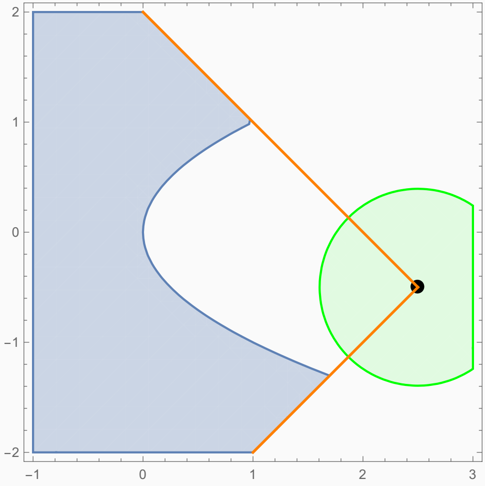
Motivation: intersection cuts

Intersection cuts in pictures



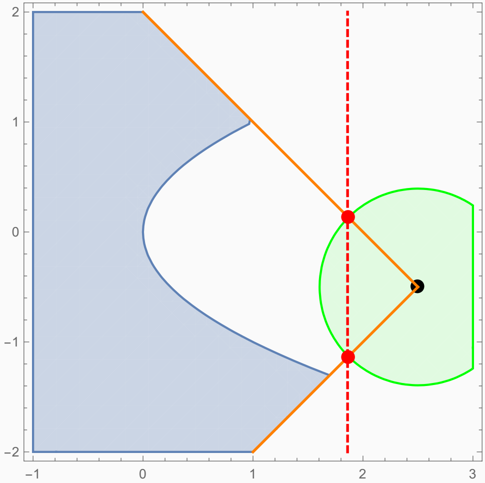
Feasible set, S (blue); $\bar{s}$ vertex of LP relaxation (black)

Intersection cuts in pictures



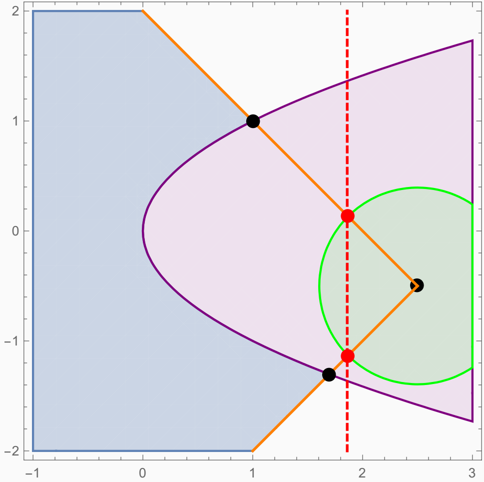
S-free set (green) (Dey and Wolsey 2010)

Intersection cuts in pictures



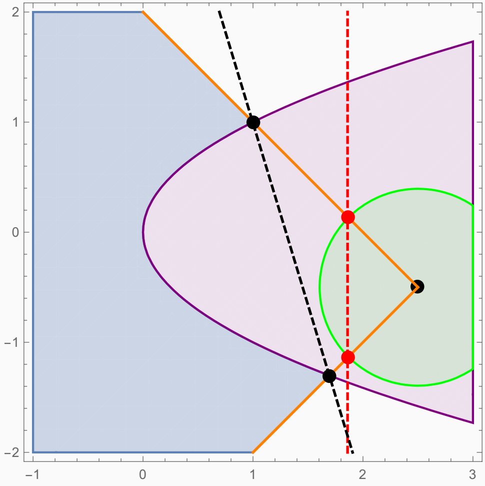
Intersection cut (red) (Balas 1971)

Intersection cuts in pictures



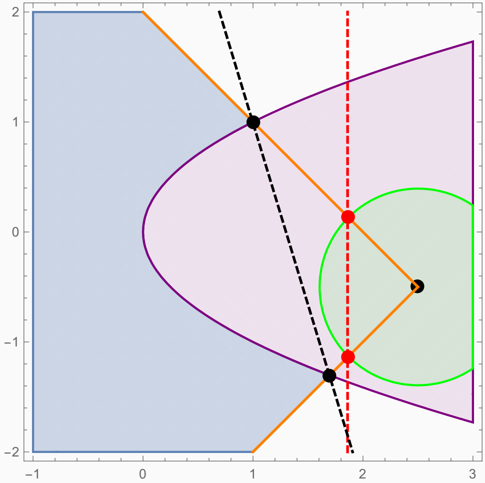
Larger S-free set (purple)

Intersection cuts in pictures



Deeper intersection cut (black)

Intersection cuts in pictures



C is *maximal S-free* if it is not contained in another S-free set

Our setting

An important case: quadratic set

$$S = \{s \in \mathbb{R}^p \ : \ s^\top A s + b^\top s + c \leq 0\}.$$

with $\bar{s} \notin S$.

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Important:

- This does not mean it only applies to problems with a single quadratic.

  $$\bar{s} \notin \bigcap_{i=1}^{m} \{s \in \mathbb{R}^p \,:\, s^\top A_i s + b_i^\top s + c_i \leq 0\}$$

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  implies there is *some* quadratic violated.
- An LP relaxation of a QCQP carries info of all constraints, thus an intersection cut would do so too.

Related work

Intersection cuts in non-convex settings

- Fischetti, Ljubić, Monaci and Sinnl (2016)$\rightarrow$ **bilevel-free sets**
- Fischetti and Monaci (2019) $\rightarrow$ **bilinear-free sets**
- Serrano (2019) $\rightarrow$ **concave underestimators of factorable functs**
- Bienstock, Chen and M. (2020) $\rightarrow$ **outer-product-free sets**
- Xu, D'Ambrosio, Liberti and Vanier (2023) $\rightarrow$ **signomial-free sets**

Beyond intersection cuts

- Kılınç-Karzan (2015) $\rightarrow$ **minimal inequalities for disjunctive conic sets**
- Burer and Kılınç-Karzan (2017) $\rightarrow$ **second-order cone intersected with quadratic**
- Santana and Dey (2018) $\rightarrow$ **convex hull of quadratic constraint $\cap$ polytope is SOC representable**

What we'll talk about today

The agenda for today: to show the basic step in the construction of

maximal quadratic-free sets

for **homogeneous quadratics** and steps toward a full characterization

Homogeneous quadratics

A canonical form for homogeneous quadratics

In this talk, we consider a set of the form

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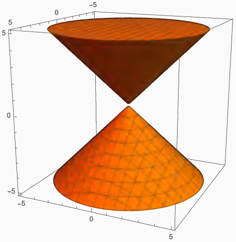
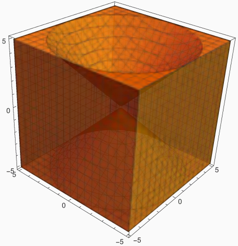


$n = 2, \; m = 1$



$n = 1, \; m = 2$

Constructing Q-free sets

$$Q = \{(x, y) \in \mathbb{R}^{n+m} \, : \, \|x\| \leq \|y\|\}$$

Since $\lambda^\mathsf{T} x \leq \|x\|$ when $\|\lambda\| = 1$, we can show that

$$C_\lambda = \{(x, y) \in \mathbb{R}^{n+m} \, : \, \|y\| \leq \lambda^\mathsf{T} x\} \quad \text{is } Q\text{-free.}$$

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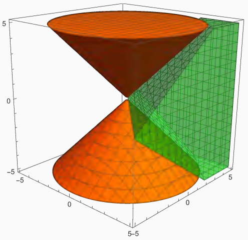
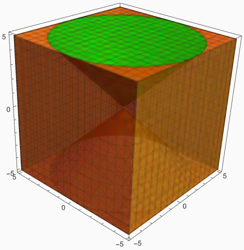
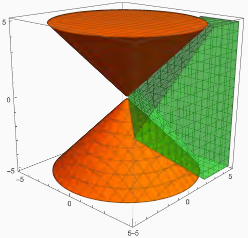
Theorem (M. and Serrano '21)

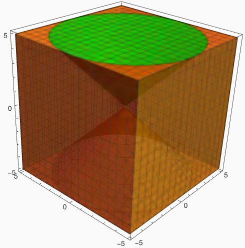
C_λ is maximal Q-free.





Maximality proof





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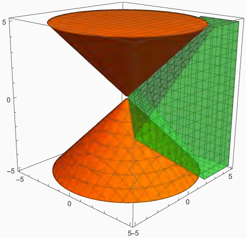

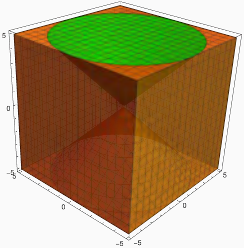

Proof sketch.
We use an outer-description of C_λ:

$$\|y\| \leq \lambda^\mathsf{T} x \Leftrightarrow \beta^\mathsf{T} y \leq \lambda^\mathsf{T} x, \ \forall \beta, \|\beta\| = 1$$

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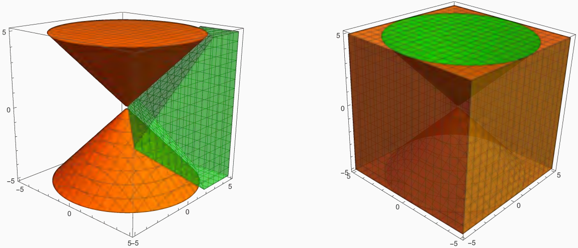

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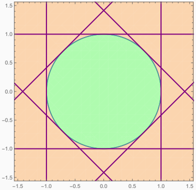
The point (λ, β) is in $Q \cap C_\lambda$ (because $\|\lambda\| = \|\beta\|$) and "exposes" the inequality $-\lambda^\mathsf{T} x + \beta^\mathsf{T} y \leq 0$. □

Having exposing points suffices for maximality



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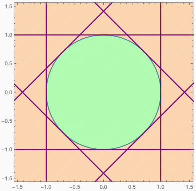

Theorem (M. and Serrano '21)

Let S be a closed set and $C = \{x \in \mathbb{R}^n : \alpha^T x \leq \beta, \forall(\alpha, \beta) \in \Gamma\}$ an S-free set. Suppose for every $\alpha^T x \leq \beta$ there is an $\bar{x} \in S \cap C$ such that

$$\underbrace{\alpha^\mathsf{T} \bar{x} = \beta \quad \textit{is the only tight inequality for } \bar{x}}_{\bar{x} \textit{ exposes } (\alpha, \beta)}$$

Then, C is maximal S-free.

This generalizes the sufficient part of the criterion of Dey and Wolsey (2010) for lattice-free sets.

Where do we go from C_λ?

Path #1: Extending C_λ to the non-homogeneous case and including integrality

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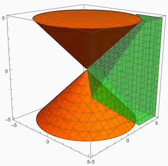
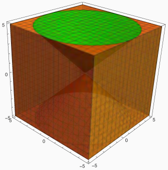
Path #2: Is C_λ all there is for Q?

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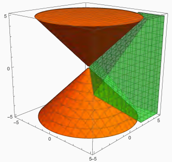
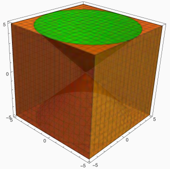
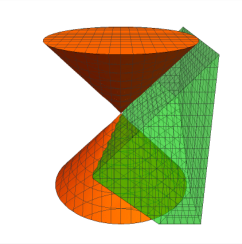




NO. The following “twisted wedge” C is also maximal:



Beyond C_λ

Rewriting Q

Recall that

$$Q = \{(x, y) \in \mathbb{R}^{n+m} \, : \, \|x\| \leq \|y\|\}$$

Since $\|y\| = \max\{\beta^\mathsf{T} y \, : \, \|\beta\| = 1\}$, we have

$$Q = \bigcup_{\|\beta\|=1} \{(x, y) \in \mathbb{R}^{n+m} \, : \, \|x\| \leq \beta^\mathsf{T} y\}$$

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$\rightarrow$ Q is the union of convex sets.

Separation of convex sets $\Rightarrow$ any Q-free set can be separated from each S_β:

$$S_\beta := \{(x, y) \in \mathbb{R}^{n+m} \;:\; \|x\| \leq \beta^\mathsf{T} y\}$$

A necessary condition for maximality

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Note that for any unit vector $\Gamma(\beta)$

$$\Gamma(\beta)^\mathsf{T} x \leq \beta^\mathsf{T} y$$

is a valid inequality for S_β. This motivates the definition of

$$C_\Gamma = \{(x,y) \in \mathbb{R}^{n+m} : \beta^\mathsf{T} y \leq \Gamma(\beta)^\mathsf{T} x \;\; \forall\, \beta \in D^m\}$$ which is always Q-free.

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We can push this idea to show

Theorem (M., Paat and Serrano '23)

Let C be a full-dimensional maximal Q-free set. There exists a function $\Gamma : D^m \to D^n$ such that

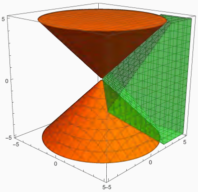
$$C = C_\Gamma.$$

Examples

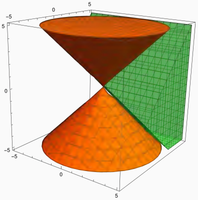
$$C_\Gamma = \{(x, y) \in \mathbb{R}^{n+m} : \beta^\top y \leq \Gamma(\beta)^\top x \ \ \forall\, \beta \in D^m\}$$

In the following 3D examples y only has one dimension $\rightarrow$ $\beta = \pm 1$.

Thus, $\Gamma(\beta)$ is part of the slopes of the two hyperplanes



$\Gamma(1) = \Gamma(-1)$



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A maximality condition

Theorem (M., Paat and Serrano '23)

If Γ satisfies that

$$\underbrace{\|\Gamma(\beta) - \Gamma(\beta')\| < \|\beta - \beta'\| \quad \beta \neq \beta'}_{\text{"strict non-expansiveness"}}$$

the set $C_\Gamma = \{(x, y) : \beta^\top y \leq \Gamma(\beta)^\top x \ \ \forall\, \beta \in D^m\}$ is maximal Q-free.

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Proof sketch.
For each β, consider the point $(x, y) = (\Gamma(\beta), \beta)$. Under the above condition

- $(x, y) \in Q \cap C_\Gamma$
- Strict non-expansiveness is equivalent to $\beta^\mathsf{T}\beta' < \Gamma(\beta)^\mathsf{T}\Gamma(\beta')$ for $\beta \neq \beta'$
- The only inequality of C_Γ which is tight at (x, y) is $\beta^\mathsf{T} y \leq \Gamma(\beta)^\mathsf{T} x$

In other words, every inequality has an exposing point

□

Γ strictly non-expansive

The simplest case of Γ strictly non-expansive is a constant function, which yields C_λ

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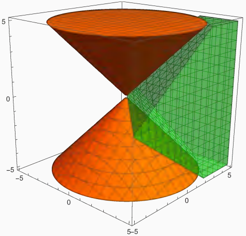
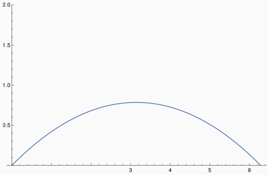


But we are not restricted to constant functions!

Γ strictly non-expansive

For example, for $n = m = 2$ we can construct a Γ function (from a circle to a circle) using polar coordinates:



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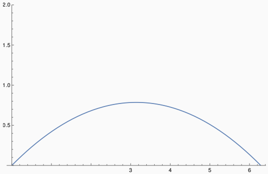
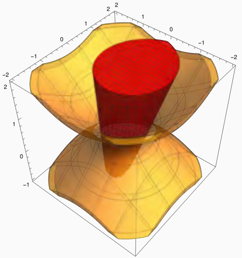


A 3D slice of the resulting 4D maximal Q-free set is:



A conjecture

We believe that the "strictly non-expansive" condition can be relaxed.

Conjecture

Consider the Q-free set

$$C_\Gamma = \{(x, y) \in \mathbb{R}^{n+m} : \beta^\mathsf{T} y \leq \Gamma(\beta)^\mathsf{T} x \quad \forall\, \beta \in D^m\}.$$

with $\Gamma : D^m \to D^n$. *If* Γ *is non-expansive, then* C_Γ *is maximal Q-free.*

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So far, we have the following partial result

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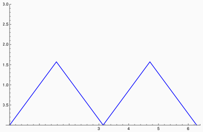
with $\Gamma : D^m \to D^n$. *If* Γ *is non-expansive and* C_Γ *is a polyhedron then* C_Γ *is maximal Q-free.*

A polyehdral example

For $n = m$ we can consider a $\Gamma(\beta) = |\beta|$. This function is non-expansive and it can be shown that it yields a polyhedral C_Γ. In polar coordinates for $n = m = 2$:



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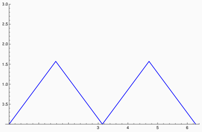

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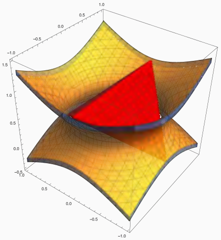
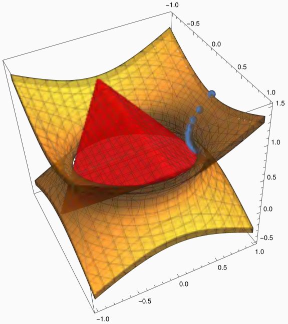


Here there's no exposing point!

Maximality proof sketch

The idea of the proof is, for each facet, to construct an exposing sequence



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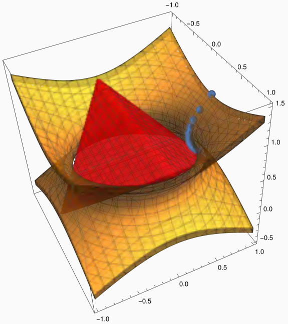
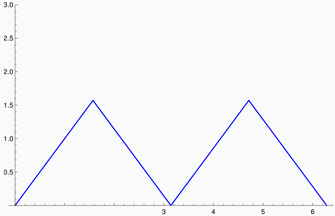


The sequence is such that every separating hyperplane sequence converges to the desired facet.

Why is this last example polyhedral?

The fact that $\Gamma(\beta) = |\beta|$ yields a polyhedral Q-free set may not be obvious. As seen before, in polar coordinates, the Γ function is



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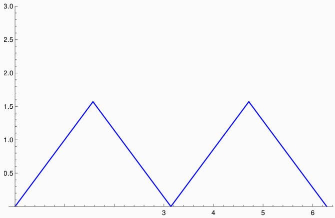

It can be shown that each break-point is a facet. Moreover, two consecutive breaking points are always isometries:

$$\|\Gamma(\beta) - \Gamma(\beta')\| = \|\beta - \beta'\|$$

and inequalities that lie "between" isometries are redundant.

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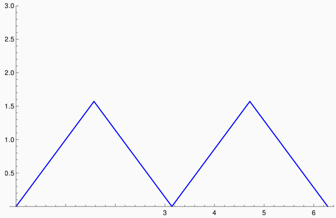


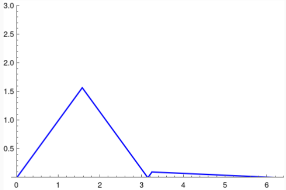
It can be shown that each break-point is a facet. Moreover, two consecutive breaking points are always isometries:

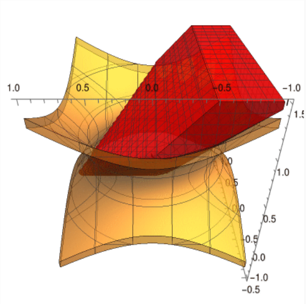
$$\|\Gamma(\beta) - \Gamma(\beta')\| = \|\beta - \beta'\|$$

and inequalities that lie "between" isometries are redundant.

In **M., Paat and Serrano (2023)** we have a full characterization of when C_Γ is a polyhedron.

What if we consider the following family of Γ functions?
(in polar coordinates)





Some last fun observations

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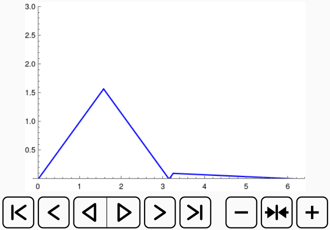

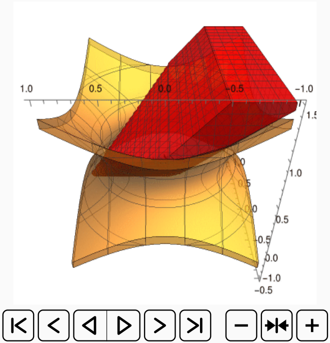

They all produce maximal Q-free sets, and only the last one is polyhedral!

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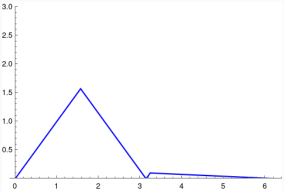

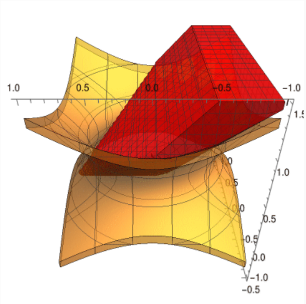

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Maximality of the non-polyhedral sets cannot be shown with the results of this talk

Summary

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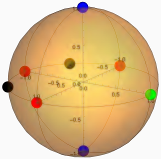
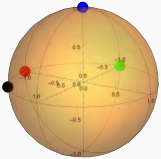
Thank you!

Bonus construction

We can use starting isometric points to construct polyhedral C_Γ sets. For instance, in 6 dimensions:



β-space



$\Gamma(\beta)$-space

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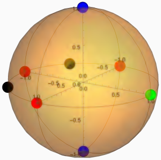

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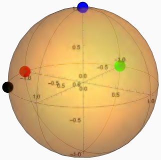

$\Gamma(\beta)$-space

This yields the following set C_Γ (3D slice)