Advances on Strictly Δ-Modular IPs

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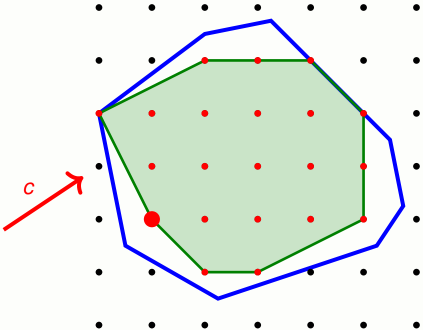
Integer Programming



Integer Linear Programming (IP)

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$, solve

$$\min\{c^\top x : Ax \leqslant b,\ x \in \mathbb{Z}^n\}\ .$$

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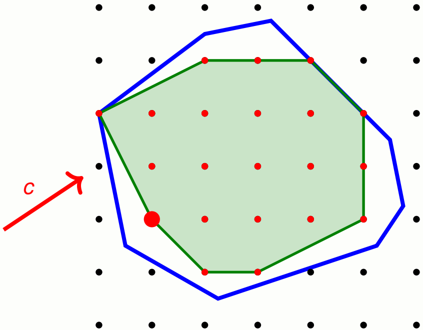

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An interesting class of efficiently solvable IPs

A totally unimodular (TU) $\implies$ Integral relaxation.

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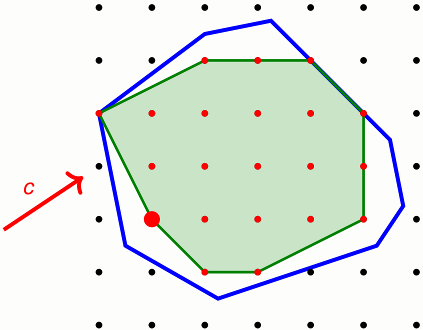

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What if minors, in absolute value, are still bounded, but not by 1?

Bounded subdeterminants

Δ-modular Integer Programming

Can IPs with Δ-modular constraint matrix be solved efficiently for constant $\Delta \in \mathbb{Z}_{>0}$?

- $A \in \mathbb{Z}^{m \times n}$ is Δ-modular if
  - $\rightarrow$ $\text{rank}(A) = n$
  - $\rightarrow$ $n \times n$ subdets bounded by Δ
- less general:
  - $\rightarrow$ total Δ-modularity: bounds on *all* subdets
  - $\rightarrow$ strict Δ-modularity: subdets in $\{0, \pm\Delta\}$ only

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Poly-time solvable special cases

✓ $\Delta = 1$: Immediate

✓ $\Delta = 2$: Bimodular Integer Programming (BIP)
[Artmann, Weismantel, and Zenklusen, STOC 2017]

✓ Totally Δ-modular IPs, at most 2 non-zeros per row
[Fiorini, Joret, Weltge, and Yuditsky, FOCS 2021]

✓ Feasibility for strictly 3-modular IPs (randomized)
[Nägele, Santiago, and Zenklusen, SODA 2022]

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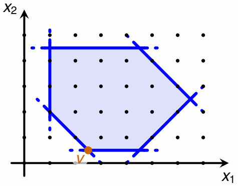
Our main result

Strongly polynomial randomized alg. for feasibility of strictly 4-modular IPs.

High-level view: BIP and strictly 3-modular feasibility



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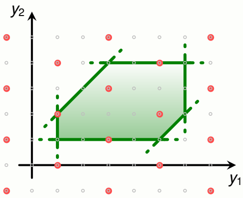

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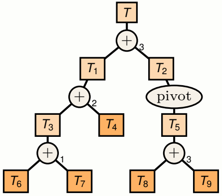
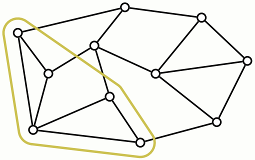
$$\min\left\{\tilde{c}^\top y\colon \begin{array}{c} Ty \leqslant b,\ y \in \mathbb{Z}^n, \\ \gamma^\top y \equiv r \pmod{m} \end{array}\right\}$$

T totally unimodular, modulus m.



Seymour's TU decomposition

Reduction to base block problems.



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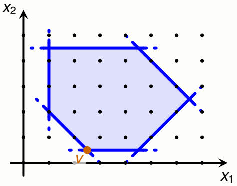

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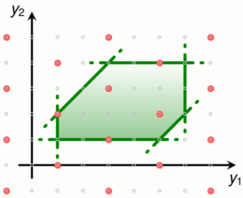

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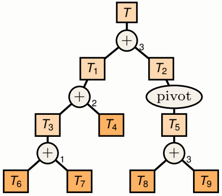

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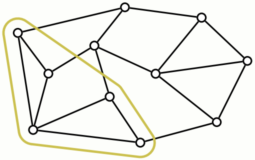

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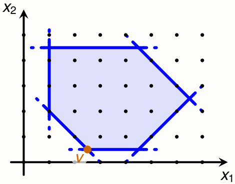

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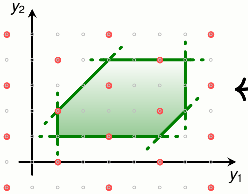

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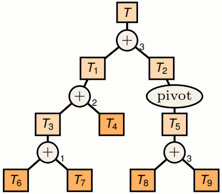
Structural results:

$\rightarrow$ proximity

$\rightarrow$ flatness or feasibility



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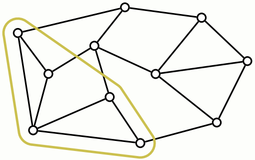

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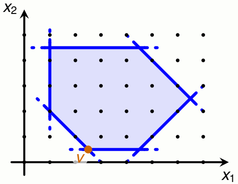

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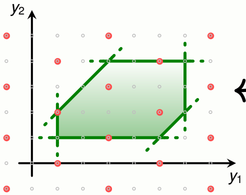

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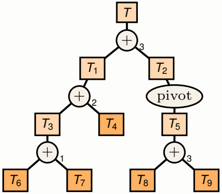

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Restriction: $m \leq 3$

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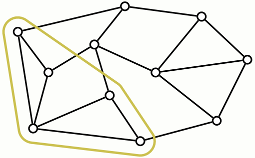

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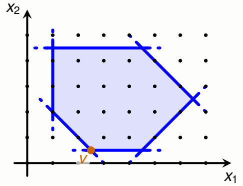

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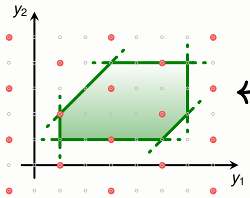

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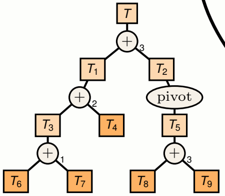

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Cut baseblock:
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Circulation baseblock:
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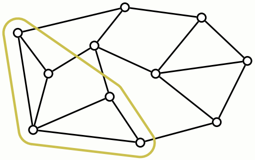
(ad hoc for $m = 2$)



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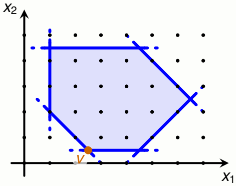

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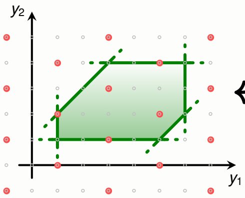

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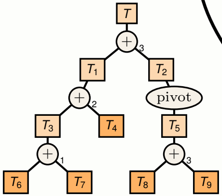

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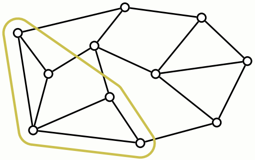

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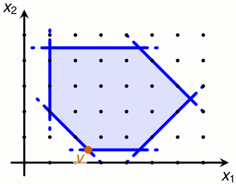

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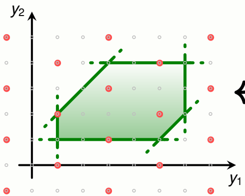

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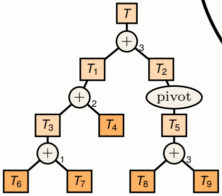

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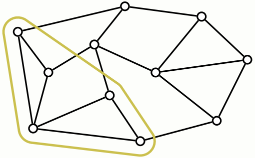

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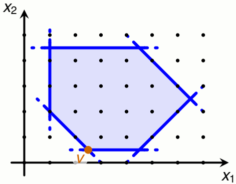

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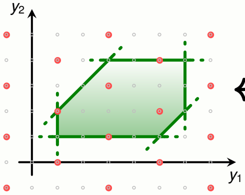

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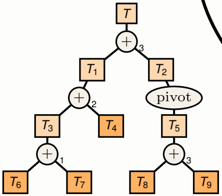

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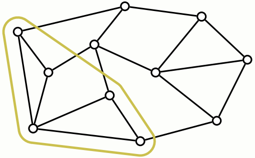

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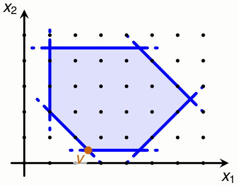

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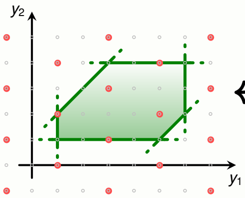

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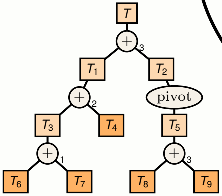

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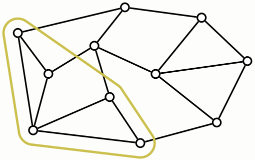

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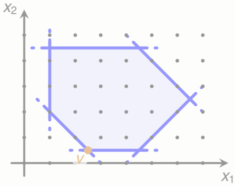

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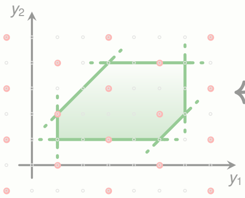

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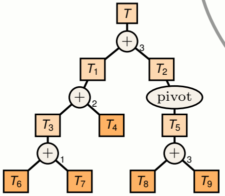

Seymour's TU decomposition

Reduction to base block problems.

Restriction: $m \leq$ ~~4~~

→ progressing in hierarchy of problems

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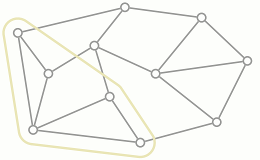
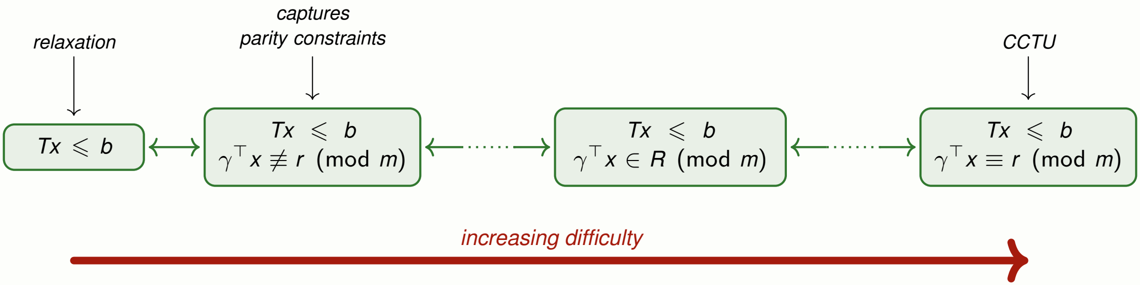


Base block problems

Interpretation as congruency-constrained cut and circulation problems

A hierarchy of congruency-constrained TU problems



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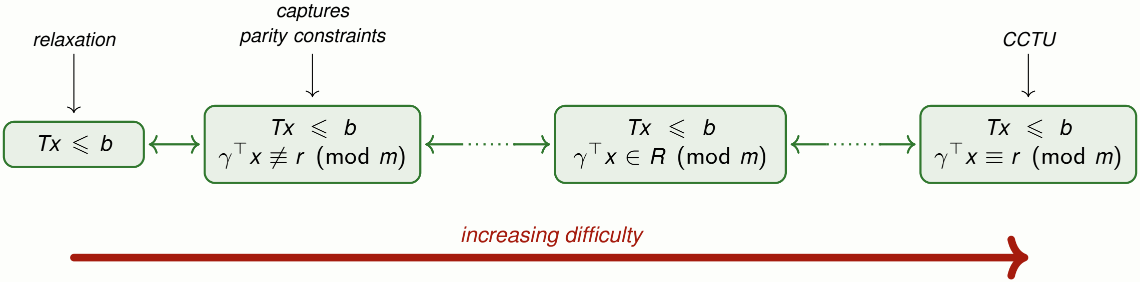


Known results

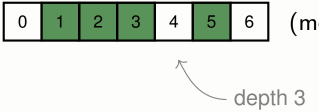
- ✓ Optimization for depth one
- ✓ Feasibility for depth two if m is prime

New result

- ✓ Feasibility for depth three and general m

Exploiting the hierarchy with Cauchy-Davenport

$$\begin{pmatrix} A & \text{rk } 1 \\ \text{rk } 1 & B \end{pmatrix} x \leqslant b$$

$\gamma^\top x \in$  (mod 7)

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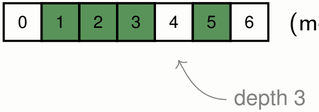

$\pmod 7$

A-subproblem

$Ax_A \leqslant b_A$

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$Bx_B \leqslant b_B$

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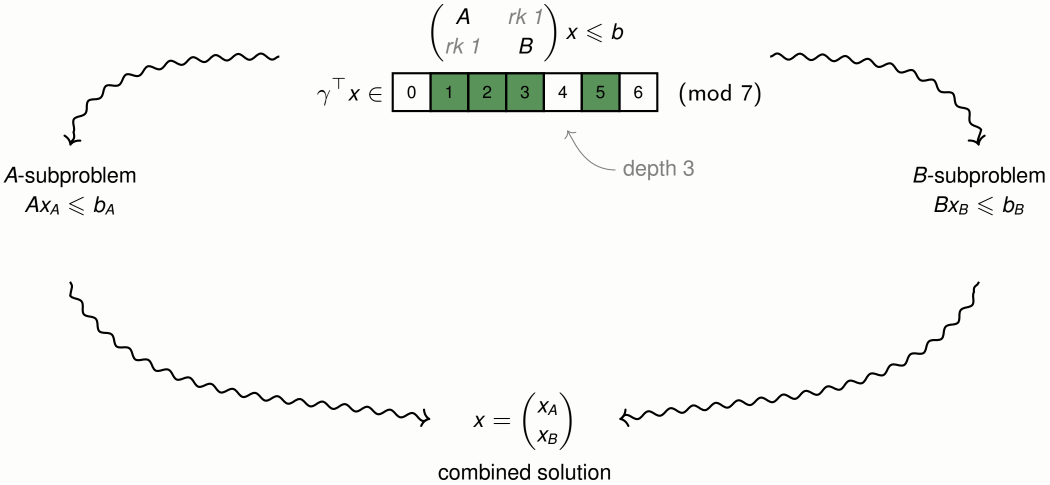

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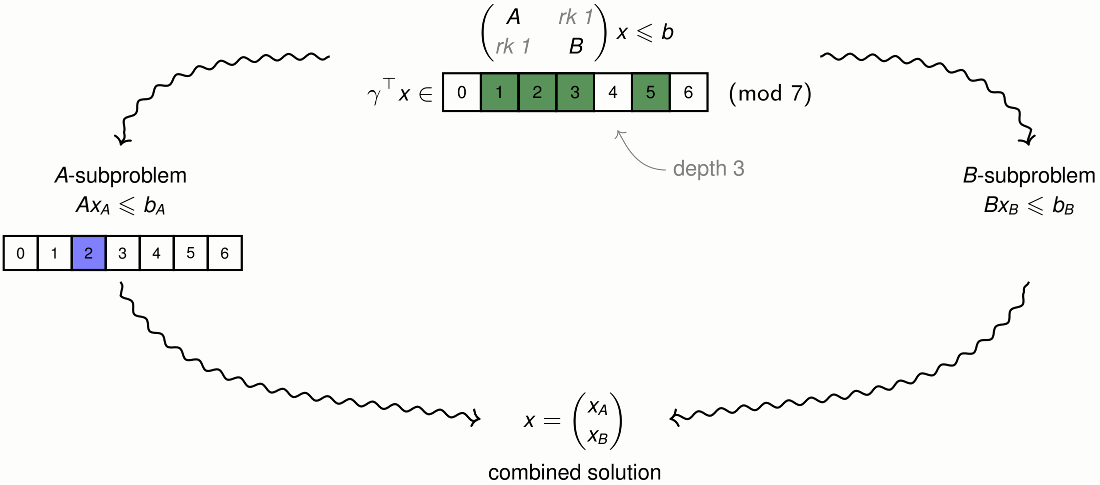

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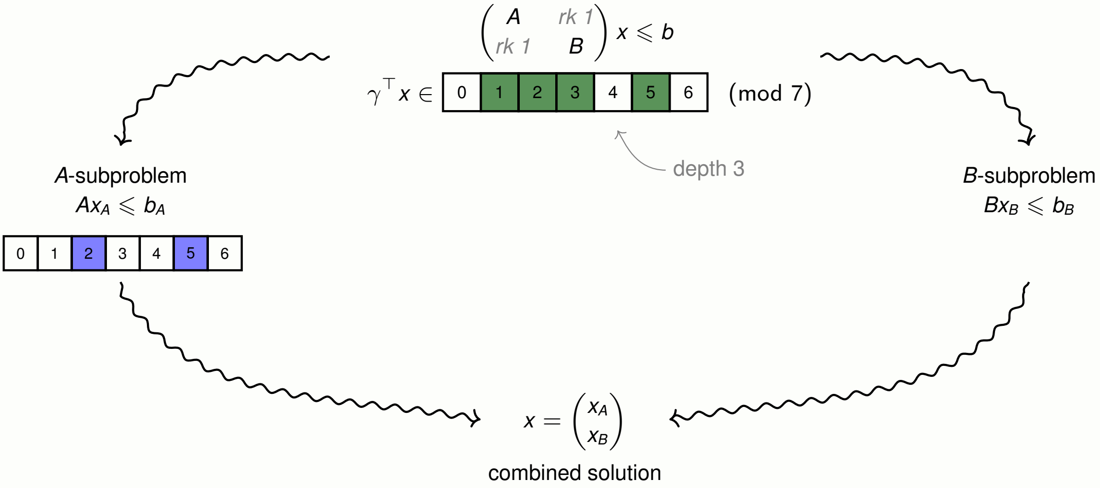

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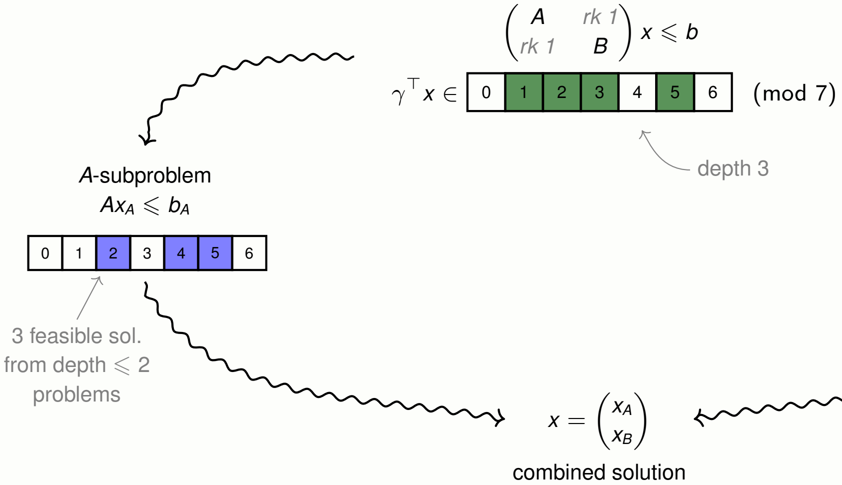

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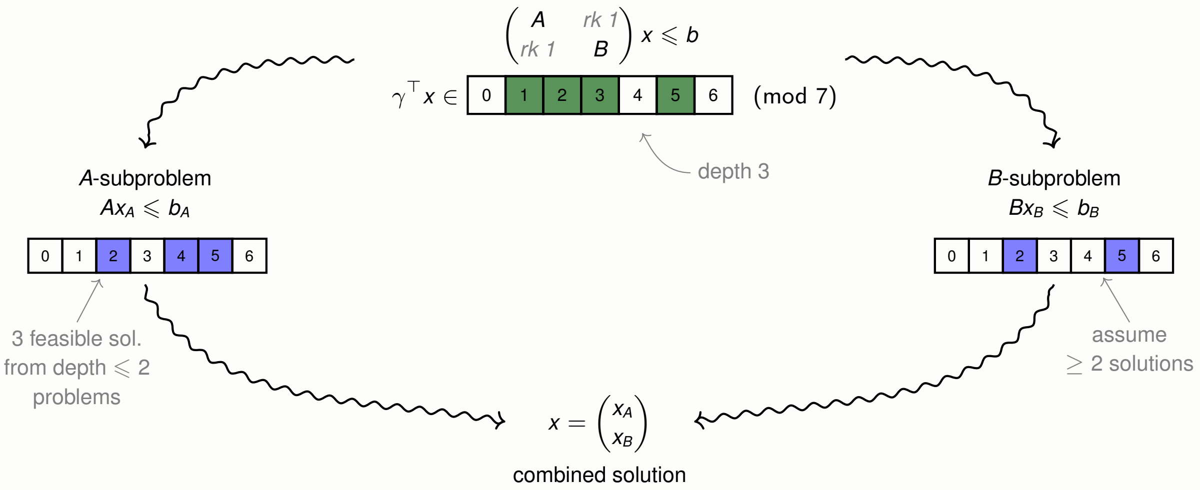

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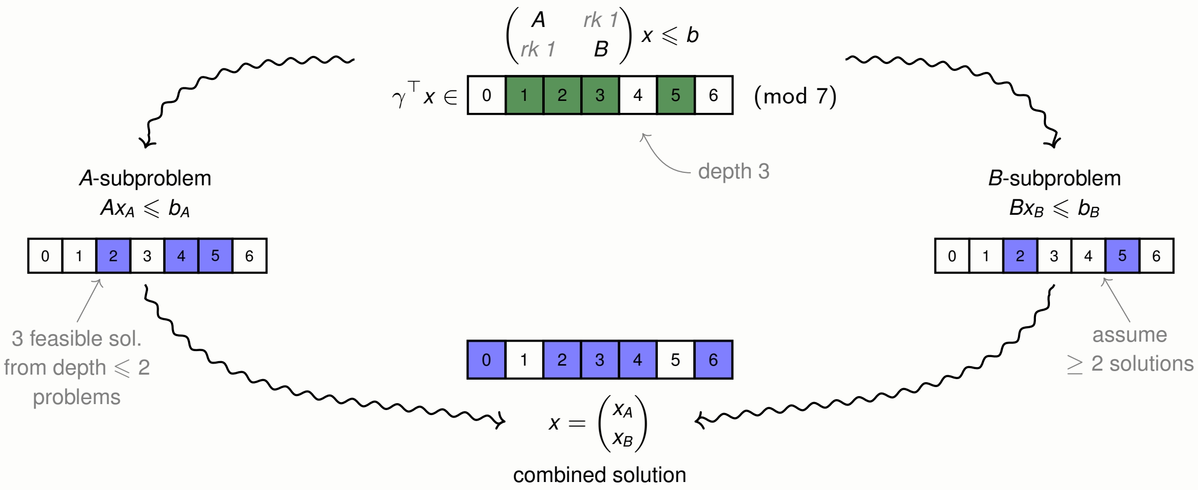

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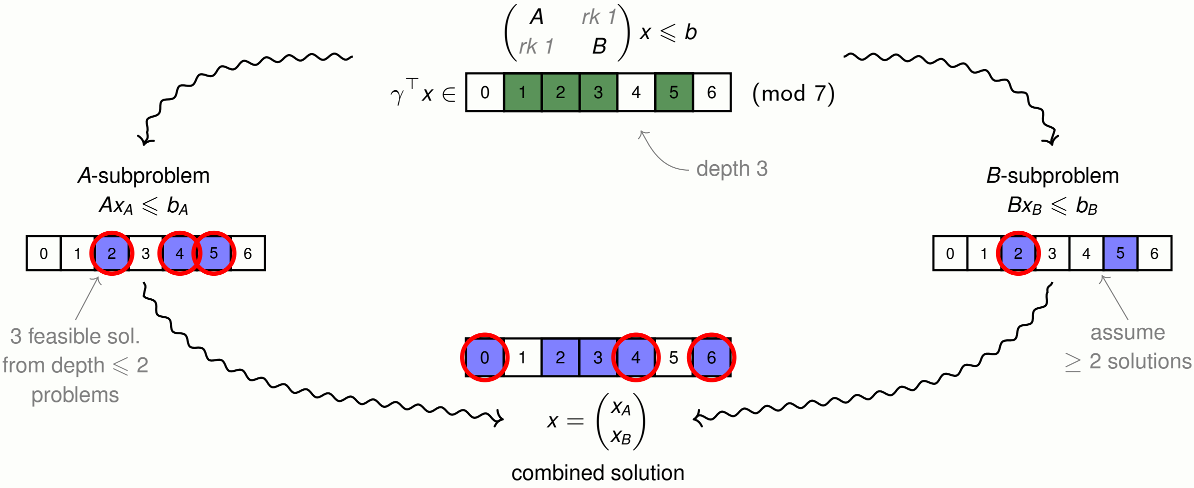

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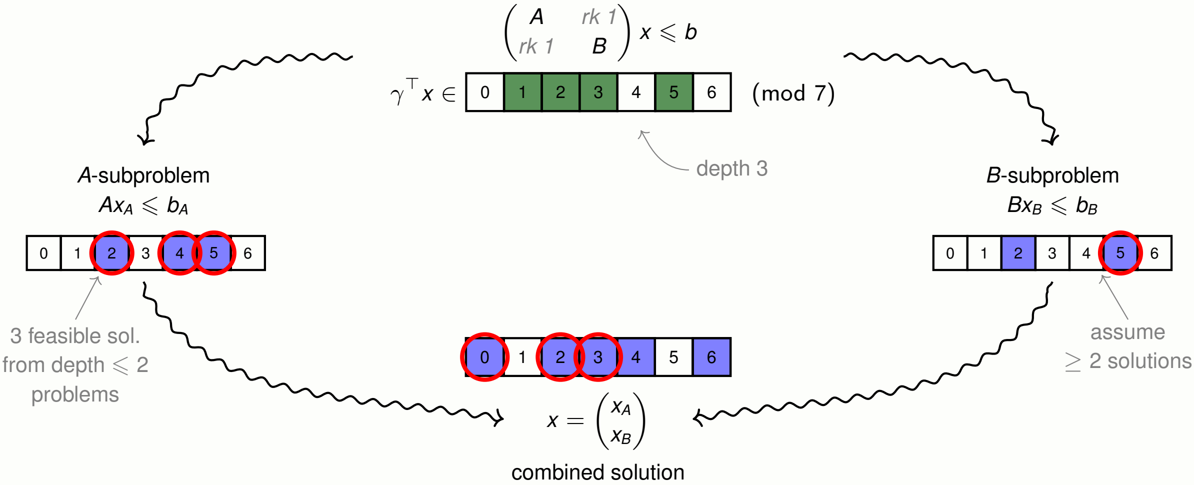

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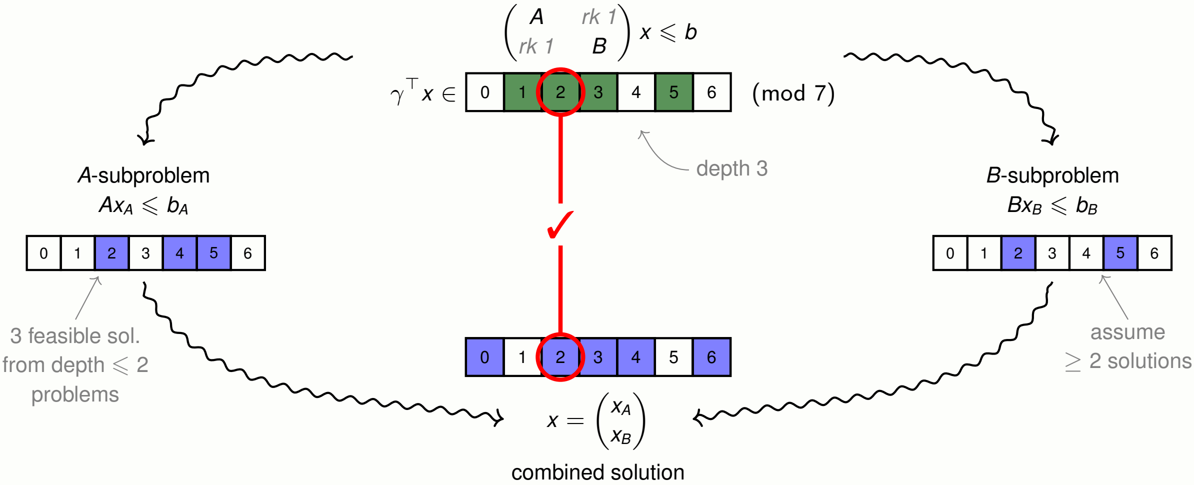

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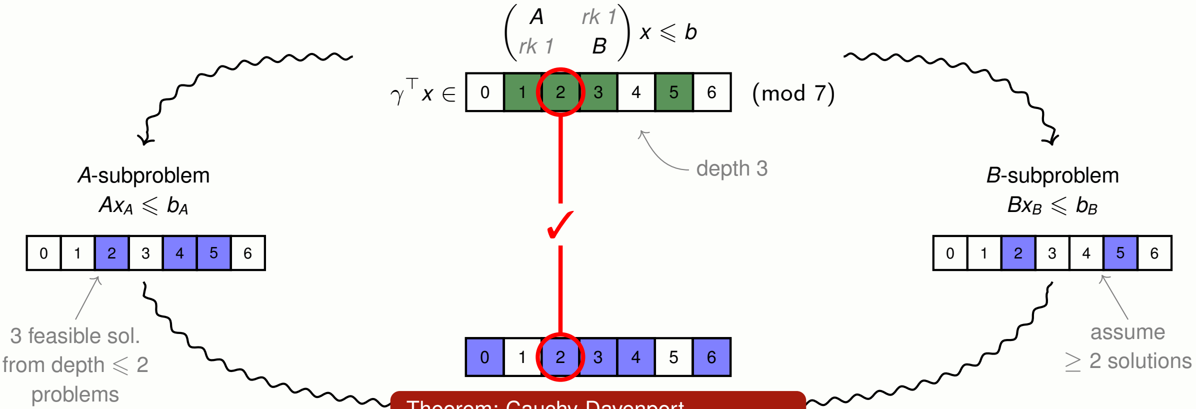
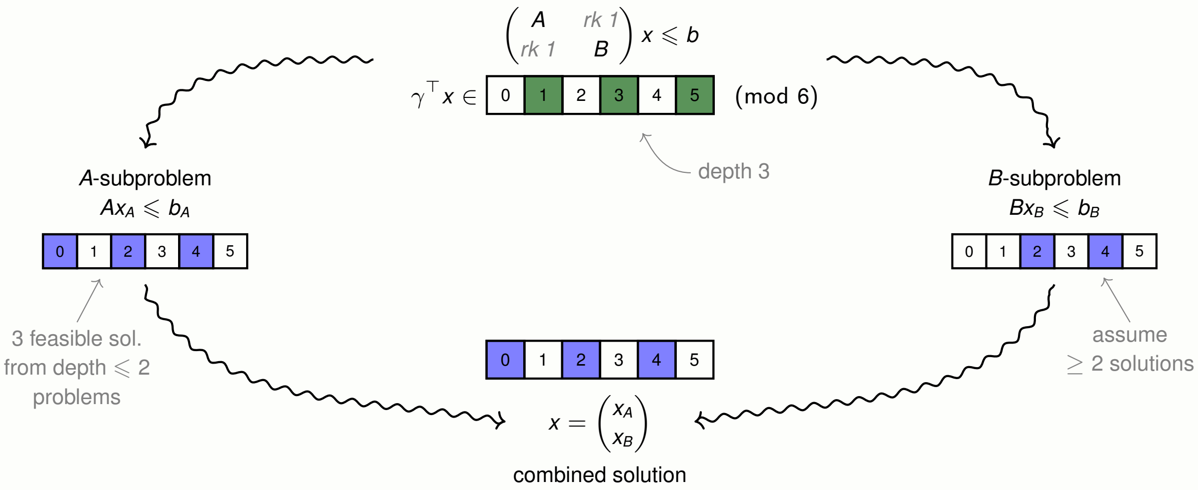


Theorem: Cauchy-Davenport

For prime m and $R_A, R_B \subseteq \mathbb{Z}_m$,
$|R_A + R_B| \geq \min\{m, |R_A| + |R_B| - 1\}$.

Non-prime modulus



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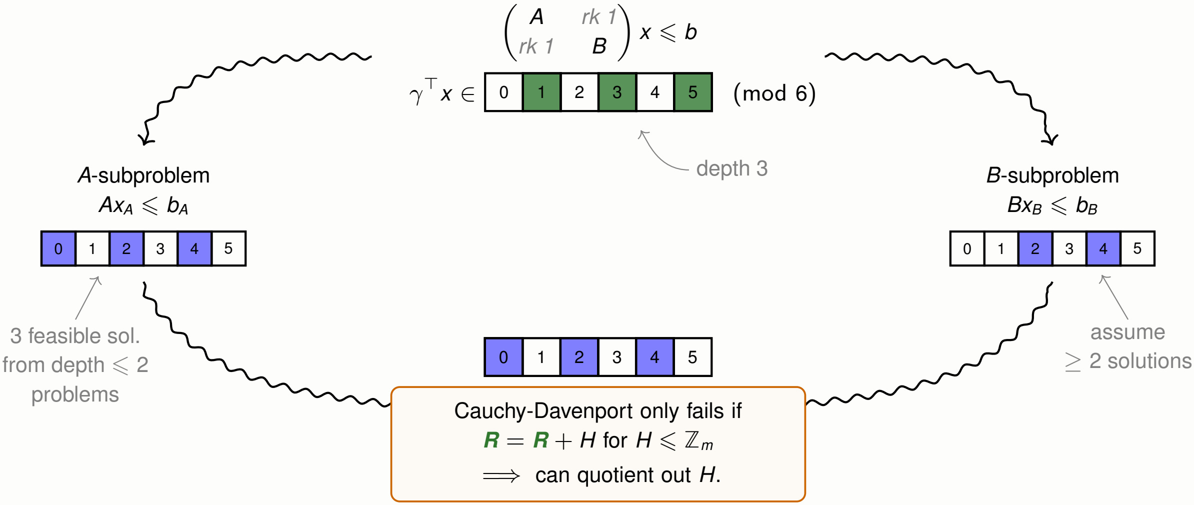


Non-prime modulus

$$\begin{pmatrix} A & \textit{rk 1} \\ \textit{rk 1} & B \end{pmatrix} x \leqslant b$$

$\gamma^\top x \in$ | 0 | **1** | 2 | **3** | 4 | **5** | (mod 6)

depth 3

$\Updownarrow$

$\gamma^\top x \in$ | 0 | **1** | (mod 2)

depth 1

A-subproblem
$Ax_A \leqslant b_A$

| 0 | 1 | 2 | 3 | 4 | 5 |

3 feasible sol. from depth $\leqslant 2$ problems

B-subproblem
$Bx_B \leqslant b_B$

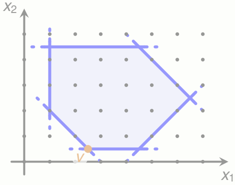
| 0 | 1 | 2 | 3 | 4 | 5 |

assume ≥ 2 solutions

| 0 | 1 | 2 | 3 | 4 | 5 |

Cauchy-Davenport only fails if
$\boldsymbol{R} = \boldsymbol{R} + H$ for $H \leqslant \mathbb{Z}_m$
$\Longrightarrow$ can quotient out H.

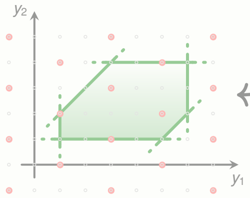
Where we are



Δ-modular integer programming

$\min\{c^\top x \colon Ax \leqslant b, x \in \mathbb{Z}^n\}$

A is Δ-modular.



Congruency-Constr. TU Prb. (CCTU)

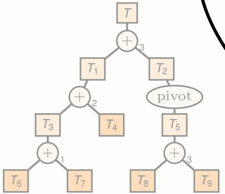
$$\min\left\{\tilde{c}^\top y \colon \begin{array}{c} Ty \leqslant b,\ y \in \mathbb{Z}^n, \\ \gamma^\top y \equiv r \pmod m \end{array}\right\}$$

T totally unimodular, modulus m.

Structural results:
→ proximity
→ flatness or feasibility

Equivalence:
$\Delta = 2$
or
A strictly Δ-modular
~~for prime Δ~~ *

*changing to multiple congruency constraints



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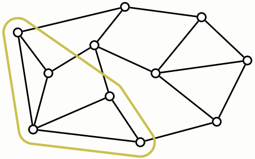

Base block problems

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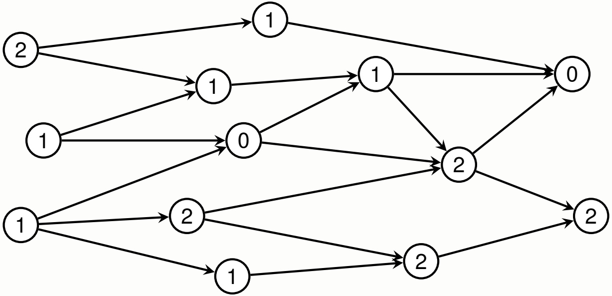
Transposed network matrices baseblock

Theorem: Transposed network matrix problems

There is a str. poly. algorithm for deciding feasibility of a congruency constraint TU problem with constant modulus and a transposed network matrix.

- Reduce to question whether a weighted lattice contains a set satisfying a congruency constraint.



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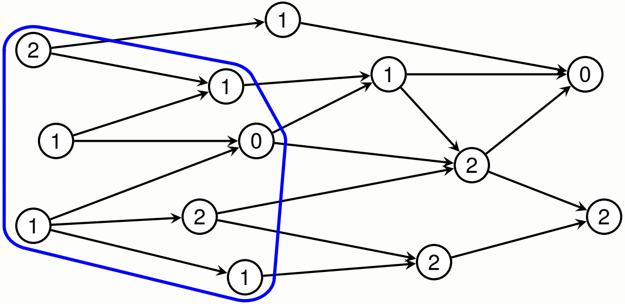
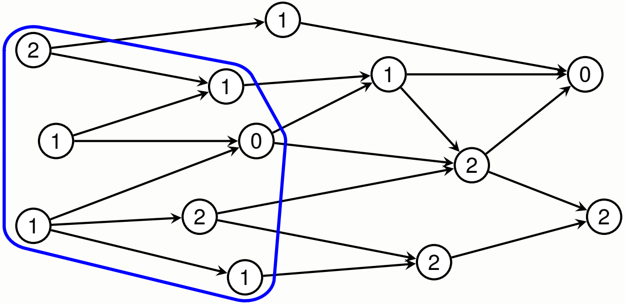
$$\gamma(C) \equiv 2 \pmod{3}$$



Solving the lattice problem



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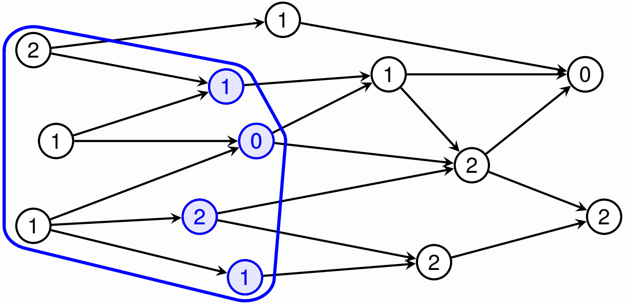

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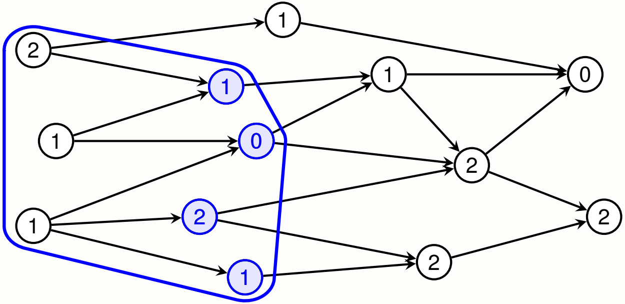
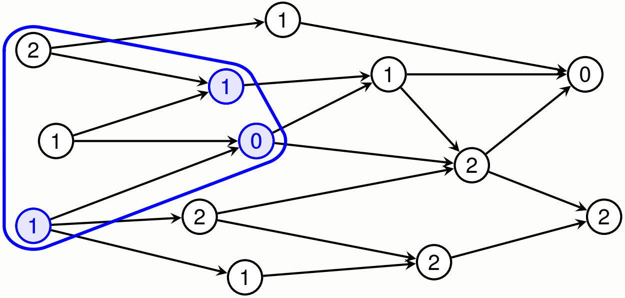


Lemma

Given m integers, there is a subset with sum divisible by m.

Solving the lattice problem



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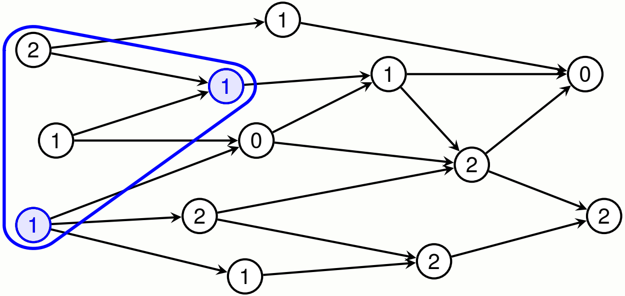

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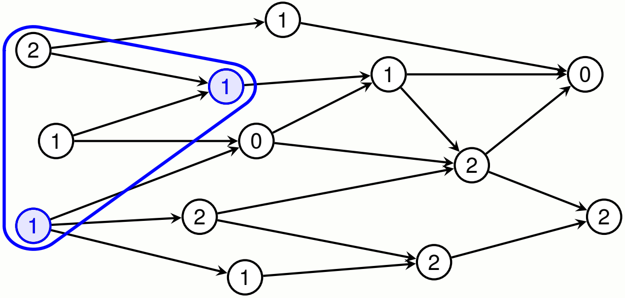

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Lemma

If there is a feasible solution, then there is one with less than m elements without successor.

Conclusions & Open Questions

- Removed prime modulus requirement in propagation

  More generally: Extension to arbitrary finite abelian groups
- Extended propagation to depth 3
- Feasibility for transposed network matrix baseblock for all moduli

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- Base blocks:

  Randomization remains necessary for congruency-constrained circulations (even feasibility)

  equivalent problem: congruency-constrained bipartite red-blue matching
- General (strictly) Δ-modular IPs
- Optimization:

  completely open beyond depth 1