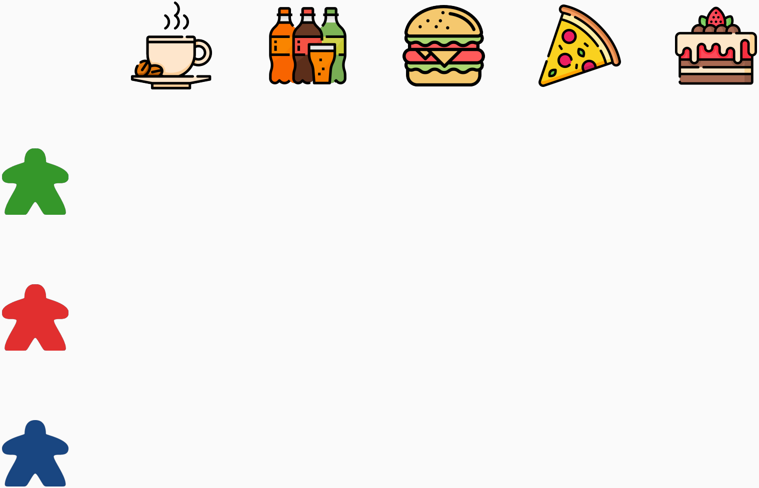
The Polyhedral Geometry of Truthful Auctions

Sylvain Spitz,
joint work w/ Michael Joswig, Max Klimm

IPCO 2023 @ Madison, Wisconsin
June 21, 2023

Allocation Mechanism



Images: flaticon.com

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			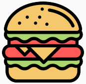		
	4.2	3.8	4.9	3.7	6.8
	5.0	5.0	5.0	5.0	5.0
	4.9	7.1	3.3	4.9	2.5

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			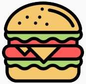		
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			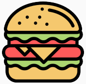		
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			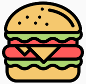		
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Allocation Mechanism

- Task: allocate m items among n agents; set of allocations:

$$\Omega \;=\; \left\{ A \in \{0,1\}^{n \times m} \;\middle|\; \sum_{i \in [n]} a_{i,j} = 1 \text{ for all } j \in [m] \right\}$$

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$$u_i(\theta' \mid \theta_i) \;=\; f_i(\theta') \cdot \theta_i - p_i(\theta')$$

- A mechanism $M = (f, p)$ is *incentive compatible* (IC), if misreporting never benefits the agent.

Difference Sets

Example (local mechanism):

One agent, two items.

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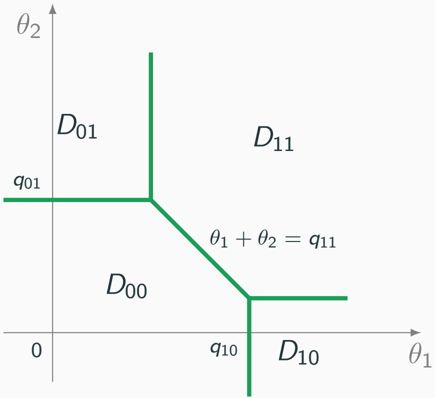
$$\max_{a \in \{0,1\}^2} \{a \cdot (\theta_1, \theta_2) - q_a\}$$



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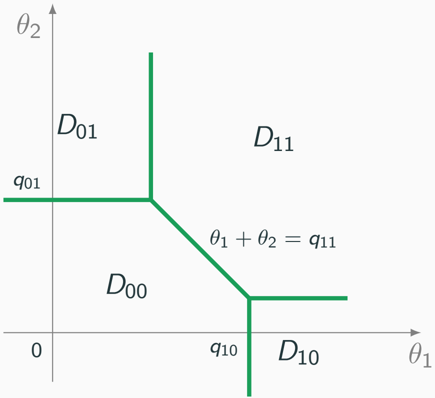
$$f(\theta) = \arg\max_{a \in \{0,1\}^2} \{a \cdot (\theta_1, \theta_2) - q_a\}$$

$$p(\theta) = q_{f(\theta)}$$



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Reduction to Single Agent

Lemma (Nisan et al. - 2007)

$M = (f, p)$ is IC if and only if for all $i \in [n]$ and all $\theta \in \mathbb{R}^{n\times m}$, p_i is given by some function $p_{i,\theta_{-i}} : \{0,1\}^m \to \mathbb{R}$, and

$$f(\theta) \in \arg\max \left\{ A_i \cdot \theta_i - p_{i,\theta_{-i}}(A_i) \;\middle|\; A \in \Omega \right\}.$$

A_i is the i-th row of the matrix A.

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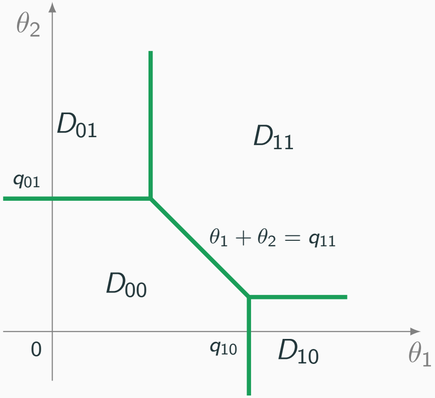
$\Rightarrow$ Multi-agent mechanisms are characterized by local one-agent mechanisms.



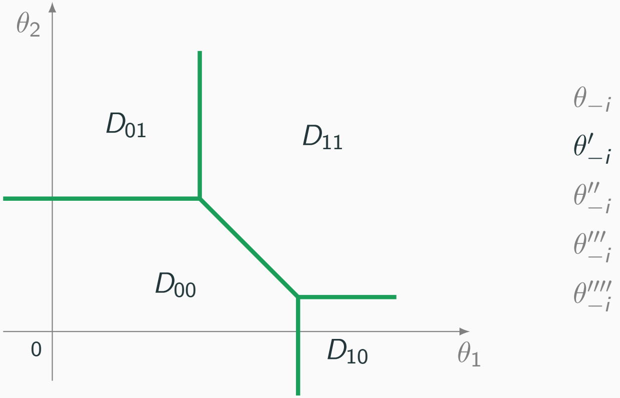
θ_{-i}

θ'_{-i}

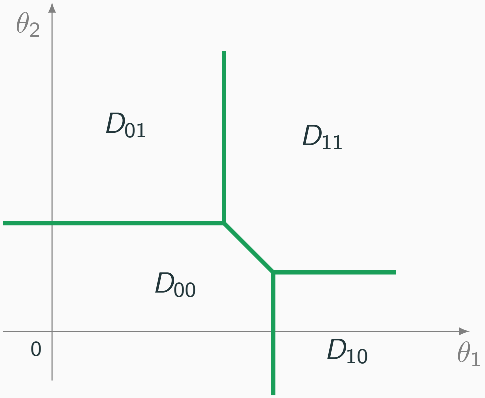
θ''_{-i}

θ'''_{-i}

θ''''_{-i}



θ_2
D_{01}
D_{11}
D_{00}
D_{10}
0
θ_1
θ_{-i}
θ'_{-i}
θ''_{-i}
θ'''_{-i}
θ''''_{-i}



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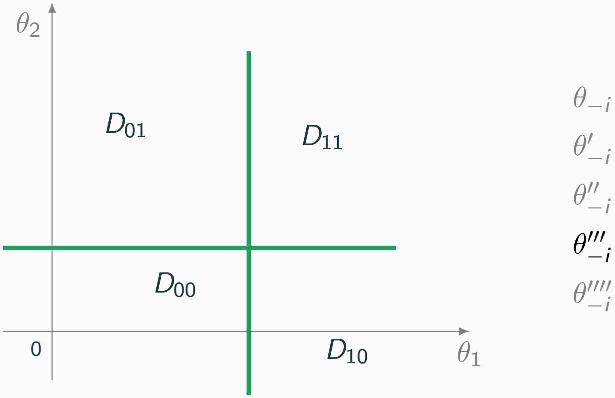
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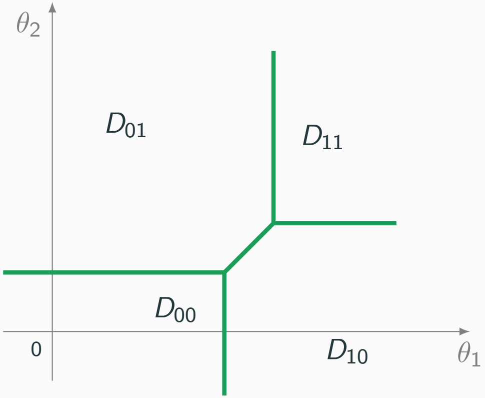

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Indifference Complex

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The *indifference complex* $\mathcal{I}(f)$ of an allocation function f is the abstract simplicial complex defined as

$$\mathcal{I}(f) \;=\; \left\{ \mathcal{O} \subseteq \Omega \;\middle|\; \bigcap_{A \in \mathcal{O}} \bar{D}_A \neq \emptyset \right\} \; .$$

$\mathcal{I}$ is an ASC $\Leftrightarrow$ (i) $\mathcal{I} \neq \emptyset$, (ii) $E \subset F, F \in \mathcal{I} \Rightarrow E \in \mathcal{I}$

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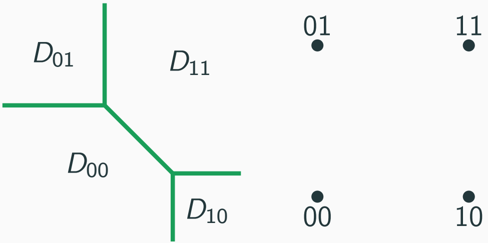

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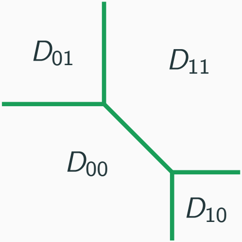

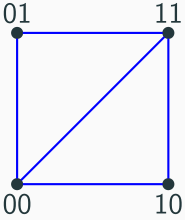

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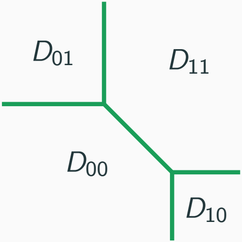

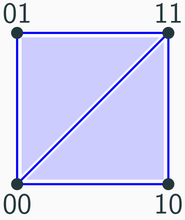

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Central Question

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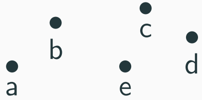
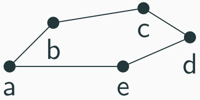
Theorem (Joswig, Klimm, S.; cf. Frongillo, Kash - 21)
An indifference complex $\mathcal{I}$ *for* m *items and one agent arises from a local IC mechanism if and only if it corresponds to a regular subdivision of the* m*-cube.*

Regular Subdivisions

Definition

Let $\mathcal{S} \subset \mathbb{R}^n$ be finite





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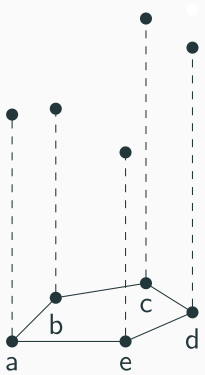

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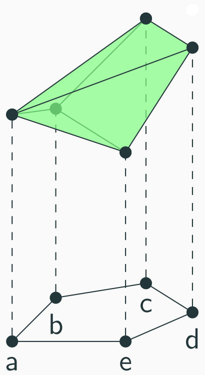

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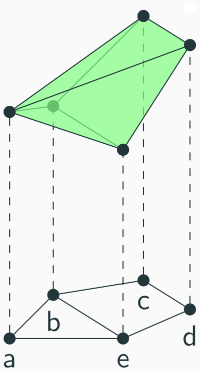

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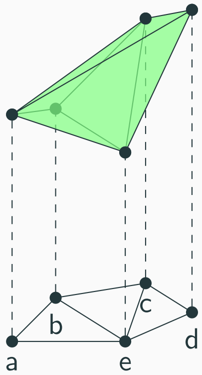

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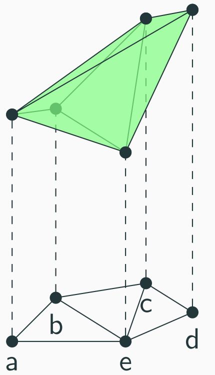

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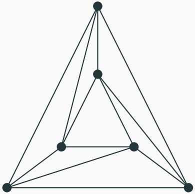
Not all subdivisions are regular, e.g.:



Number of IC Mechanisms

Theorem (Joswig, Klimm, S.; cf. Frongillo, Kash - 21)

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- A mechanism is nondegenerate, if the associated regular subdivision is a triangulation.
- Number of triangulations of the m-cube:

m	all	regular
2	2	2
3	74	74
4	92,487,256	87,959,448

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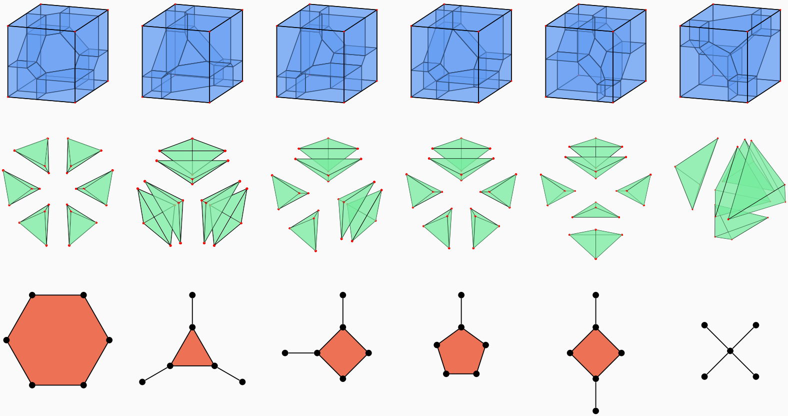
m	all	regular	S_m-orbits	Γ_m-orbits
2	2	2	2	1
3	74	74	23*	6
4	92,487,256	87,959,448	3,706,261*	235,277

*Computations made using MPTOPCOM

Γ_3-Orbits



Type A (4 reg, 2 S_3) | Type B (8 reg, 4 S_3) | Type C (24 reg, 6 S_3) | Type D (24 reg, 6 S_3) | Type E (12 reg, 3 S_3) | Type F (2 reg, 2 S_3)

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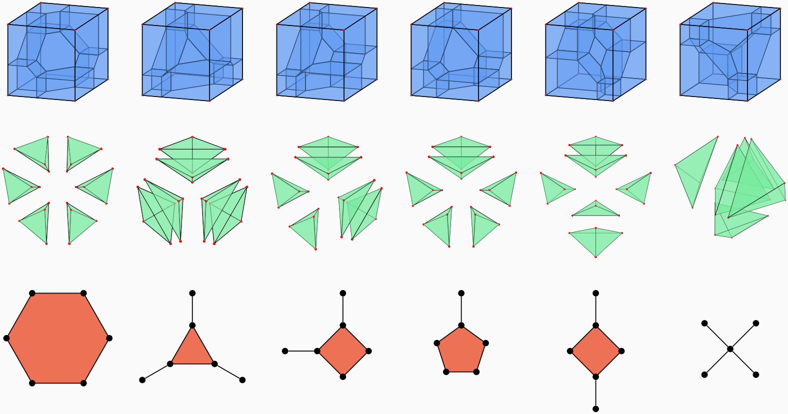

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- Type A – E have been found by Vidali(2009)

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- Hamming distance: $d_h(a,b) = |a-b|_1$ $\quad\to \mu_h(f)$

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- Hamming distance: $d_h(a,b) = |a-b|_1$ $\quad \to \mu_h(f)$
- What is $M_c(m) = \min_{f \in \Phi_m} \mu_c(f)$? (Resp. $M_h(m)$?)

  Φ_m = set of local allocation functions for m items

Minimal Sensitivities

Proposition (Joswig, Klimm, S.)

The minimal cardinality sensitivity of an IC single agent mechanism for m items is $M_c(m) = 1$.

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The minimal Hamming sensitivity of an IC single agent mechanism for $m \geq 3$ *items is bounded by* $2 \leq M_h(m) \leq m - 1$.

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The minimal cardinality sensitivity of an IC single agent mechanism for m items is $M_c(m) = 1$.

Proof. Cut the cube with the hyperplanes

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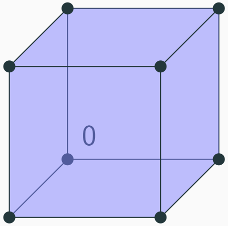

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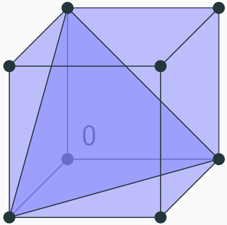

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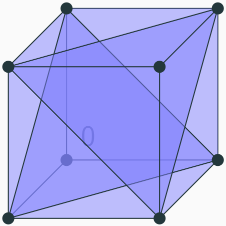

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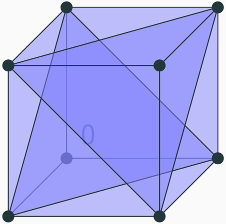

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Proof. Upper bound, m odd: Cut off all corners with even number of ones $\Rightarrow$ no antipodal vertices in the same cell.

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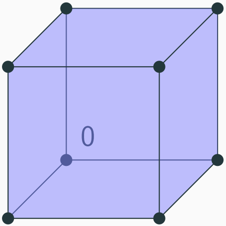

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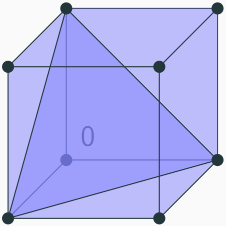

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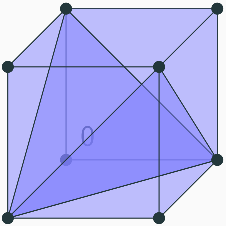

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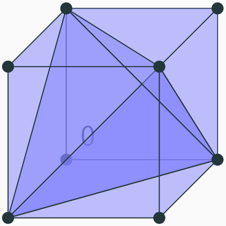

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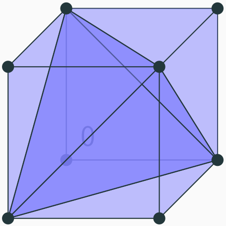

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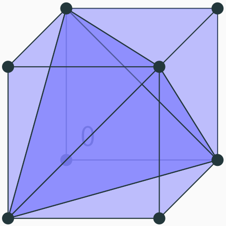

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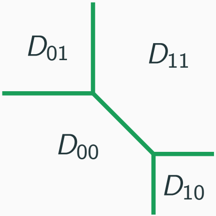
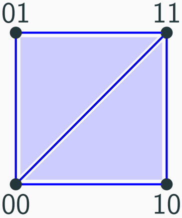
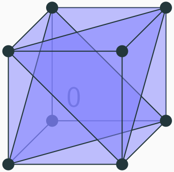
Thank You for Your attention!







Affine Maximizers

Allocation space for n agents and m items:

$$\Omega \;=\; \left\{ A \in \{0,1\}^{n\times m} \;\middle|\; \sum_{i\in[n]} A_{i,j} = 1 \text{ for all } j \in [m] \right\}$$

f is an *affine maximizer* $\Leftrightarrow$ There exist $w_1, \ldots, w_n \in \mathbb{R}$ and $c_A \in \mathbb{R}$ for all $A \in \Omega$, such that

$$f(\theta) \;\in\; \arg\max \left\{ c_A + \sum_{i\in[n]} w_i \theta_i \cdot A_i \;\middle|\; A \in \Omega \right\} \;.$$

$\Omega =$ vertex set of $\Delta_{n-1}^m \quad (\Delta_{n-1} = \text{conv}\{e_1, \ldots, e_n\})$

Indifference Complexes of Affine Maximizers

Affine maximizer:

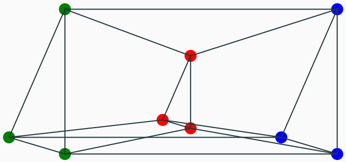
$$f(\theta) \;\in\; \arg\max\left\{ c_A + \sum_{i\in[n]} w_i\theta_i \cdot A_i \;\middle|\; A \in \Omega \right\} \;.$$

Theorem (Joswig, Klimm, S.)
An indifference complex $\mathcal{I}$ for n agents and m items arises from an affine maximizer if and only if it corresponds to a regular subdivision of Δ^m_{n-1}.

Symmetries of Δ_{n-1}^m

$$\Omega \; = \; \left\{ A \in \{0,1\}^{n \times m} \;\middle|\; \sum_{i \in [n]} A_{i,j} = 1 \text{ for all } j \in [m] \right\}$$

- Regular subdivisions of Δ_{n-1}^2 have been studied before.



- Denote by $S_n \times S_n$ the automorphism group which permutes the vertices of each simplex separately.
- Denote by $S_m \times S_n$ the automorphism group which permutes the rows and columns of allocations $A \in \{0,1\}^{n \times m}$.

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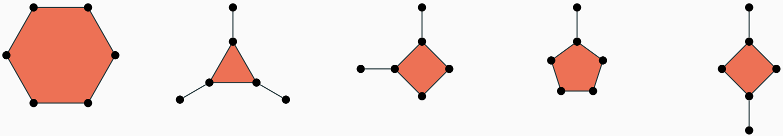
Results for $m = 2$:

n	regular	$[S_2 \times S_n]$-orbits	$[S_n \times S_n]$-orbits
3	108	21	5
4	4,494,288	96,722	7,869

Computations made using MPTOPCOM

Triangulations of Δ_2^2



Type A	Type B	Type C	Type D	Type E
6 regular	12 regular	36 regular	36 regular	18 regular
3 $S_3 \times S_3$	4 $S_3 \times S_3$	5 $S_3 \times S_3$	5 $S_3 \times S_3$	4 $S_3 \times S_3$

Sensitivity of Affine Maximizers

Cardinality distance: $d_c(a, b) = \Big| |a|_1 - |b|_1 \Big|$. The cardinality sensitivity of an affine maximizer f is

$$\mu_c(f) = \max \{ d_c(A_i, B_i) \mid A, B \in F \text{ for some } F \in \mathcal{I}(f) \text{ and } i \in [n] \}$$

Proposition (Joswig, Klimm, S.)

The minimal cardinality sensitivity of affine maximizers for $n \geq 3$ agents and m items is bounded by $\mu_c(f) \leq \lceil \frac{m}{2} \rceil$.

This sensitivity can be achieved by the allocation biases

$$c_A = -\max_{i \in [n]} \left(\sum_{j \in [m]} a_{i,j} \right)^2$$