Towards an Optimal Contention Resolution Scheme for Matchings

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Overview

1. Introducing contention resolution schemes
2. Previous literature
3. Our contributions
4. Key ideas
5. Open questions

Contention resolution schemes in the simplest setting[1]

- Imagine n people all of whom want access to a resource.
- Each person requests access to the resource independently, with probability p_i. We will assume $\sum_{i=1}^{n} p_i = 1$.
- The resource can be allocated to only one person.
- The resource must be allocated fairly, i.e, the chance of a person's request being accepted, given they make the request, is the same for all people. In other words, there is a constant c such that the i^{th} person's request is accepted with probability cp_i.

[1] Feige 2009.

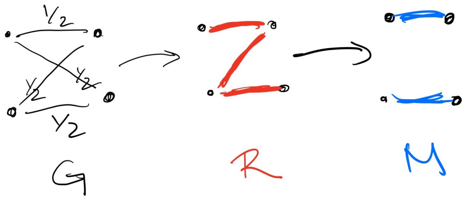
Contention resolution schemes in the simplest setting

- It is possible to prove that we can allocate to person i with probability $\left(1 - \frac{1}{e}\right) p_i$, but we can't do better (when all the p_i equal $\frac{1}{n}$).

Contention resolution schemes for matchings

- A graph G with set of edges E.
- Each edge e appears independently with probability p_e in a random graph R. The vector (p_e) is a fractional matching.
- Amongst the edges in R, can only accept at most one edge at each vertex, i.e., can only accept a matching $M \subset R$.
- Chance that an edge e is accepted given it appears in R is the same for all edges e, i.e, $\Pr[e \in M] = cp_e$.

What is the largest c we can achieve (for the worst possible choice of p_e)?



Contention resolution schemes for matchings

- Roughly speaking, find as large a matching as you can in a random graph, while being fair to every edge.

Applications of contention resolution schemes

- Combinatorial allocation problems.
- Combinatorial optimization through randomized rounding[2] and correlation gap.
- For the online version, prophet inequalities and sequential pricing problems.

[2] Chekuri, Vondrák, and Zenklusen 2014.

Previous literature

- Upper bound of $\simeq 0.544$ due to Karp and Sipser[3]. (They prove this upper bound without the fairness constraint.)
- Lower bound of $\simeq 0.476$ for bipartite matchings due to Bruggmann and Zenklusen[4]. This is the optimal *monotone* scheme.
- Lower bound of 0.474 for general matchings due to MacRury, Ma, and Grammel (actually assumes the edges appear in a random-order online fashion)[5].

[3] Karp and Sipser 1981.
[4] Bruggmann and Zenklusen 2022.
[5] MacRury, Ma, and Grammel 2022.

Our contributions

Theorem 1

There is a contention resolution scheme with selectability $\simeq 0.544$ *for general matchings, as long as the* p_e *are all small.*

Theorem 2

There is a contention resolution scheme with selectability $\simeq 0.509$ *for bipartite matchings, beating the best possible monotone scheme, and the best possible random order online scheme.*

Contention resolution schemes for matchings

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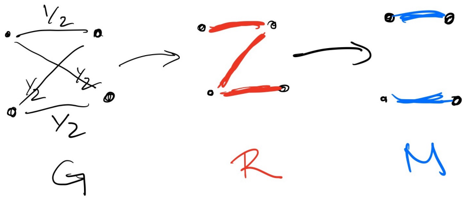


Algorithm[6] for Theorem 1

Theorem 1

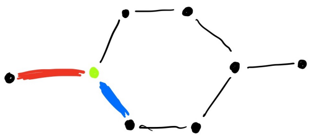
There is a contention resolution scheme with selectability $\simeq 0.544$ for general matchings, as long as the p_e are all small.

- Add a random edge adjacent to a degree 1 “leaf” vertex to the matching. Remove both vertices from the graph and repeat.

[6] Karp and Sipser 1981.

Key Idea 1 behind Theorem 1

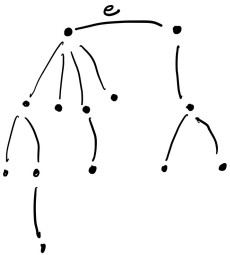
- It is not harmful to add an edge adjacent to a leaf to a matching.



- Either green vertex is included or not in a matching. If included, can swap blue edge for red edge if necessary. If not, can add red edge, making matching bigger.

Key Idea 2 behind Theorem 1

- When the p_e are small, the graph R, in the neighbourhood of any edge e, looks like a random tree.



- At each vertex, the probabilities of the edges adjacent to it sum to 1. If all the probabilities are small, it is reasonable to say that each vertex has Poi(1) many children.

Key Idea 2 behind Theorem 1

- Analyzing the performance of the Karp-Sipser algorithm is easier on random trees than on R.
- While the algorithm is due to Karp and Sipser, they only studied the expected size of the maximum matching. We simplify and generalize the analysis to also prove that the fairness constraint is met.

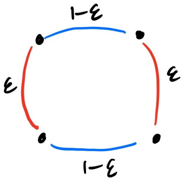
Theorem 2

Theorem 2

There is a contention resolution scheme with selectability $\simeq 0.509$ *for bipartite matchings, beating the best possible monotone scheme, and the best possible random order online scheme.*

Key Idea 1 behind Theorem 2

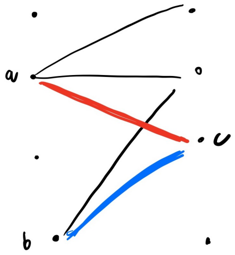
- When performing contention resolution it is often useful to reduce the probability of appearance of the edges most likely to appear.



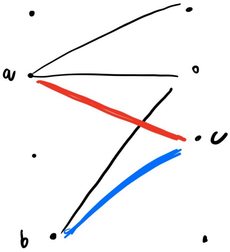
- If we aim for a large matching, the blue edges almost always appear, and we will select them, stopping us from being fair to the red edges. To combat this, it is useful to pretend as if the blue edges actually appear with a lower probability.

Key Idea 2 behind Theorem 2

- For bipartite graphs, it is useful to run contention resolution schemes in two stages–first selecting an edge at each vertex on the left, and then choosing an edge amongst selected edges at each vertex on the right.



Key Idea 2 behind Theorem 2



- In the first stage, we perform contention resolution at vertex a, and select the red edge. At vertex b, we select the blue edge. In the second stage, we have to run a contention resolution scheme to choose between the red and blue edges at vertex c.

Key Idea 3 behind Theorem 2

- It is not harmful to add an edge adjacent to a leaf to a matching.

Algorithm sketch for Theorem 2

Theorem 2

There is a contention resolution scheme with selectability $\simeq 0.509$ *for bipartite matchings, beating the best possible monotone scheme, and the best possible random order online scheme.*

1. Reduce the probabilities of the highest probability edges.
2. Perform contention resolution at each vertex on the left, prioritizing picking edges adjacent to leaves.
3. Perform contention resolution at each vertex on the right amongst the edges picked at the previous stage.

Open Questions

- Find the optimal contention resolution scheme for bipartite matchings and general matchings. Ideas...
  - Find a reduction to the small p_e case (at least for bipartite matchings) to prove that the optimal factor for the problem is really $\simeq 0.544$.
  - Prove that the worst case occurs when all the edge probabilities are equal (analogous to Van der Waerden's conjecture).
- Investigate the online version of the problem more thoroughly.

Thank You

References

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