From approximate to exact integer programming

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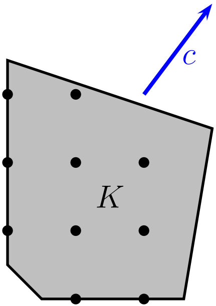
Integer programming

$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{Z}^n\}$$



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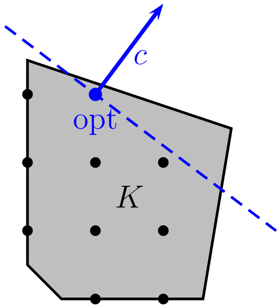

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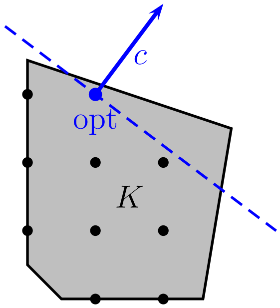

Applications:

- Logistics / transportation
- Scheduling / timetabling
- Chip design
- Inventory management

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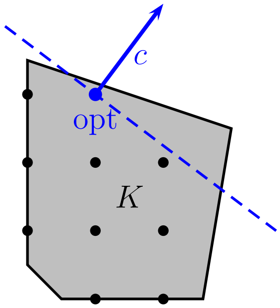

Mathematics at intersection of

- Convex geometry
- Geometry of numbers
- Convex optimization

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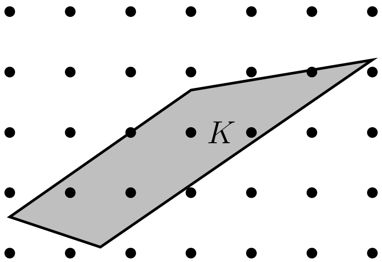
- Improved to $n^{O(n)}$-time algorithm [Kannan 1983]
- Very different $2^{O(n)} n^n$-time algorithm by [Dadush 2012] (rather involved, following [Kannan, Lovász 1988])

Currently best Lenstra-type bound: $(\tilde{O}(n^{4/3}))^n$

Modern interp. of Lenstra's algorithm



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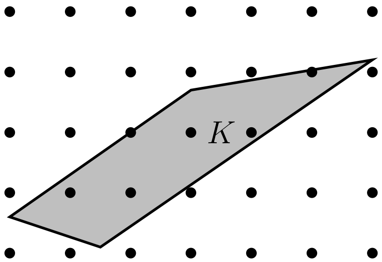

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  $\mu(\mathbb{Z}^n, K) \cdot \lambda_1(\mathbb{Z}^n, (K-K)^\circ) \leq \tilde{O}(n^{4/3})$

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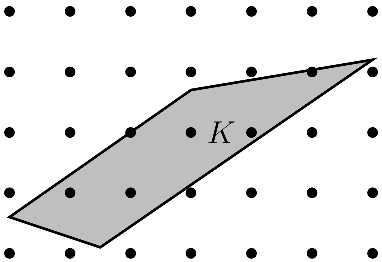

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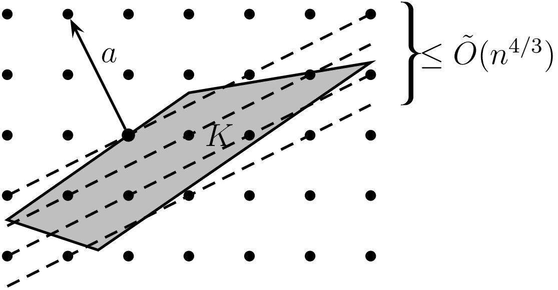

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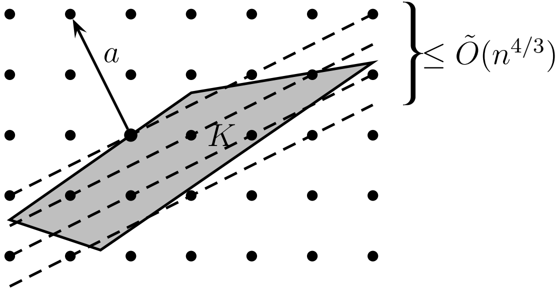

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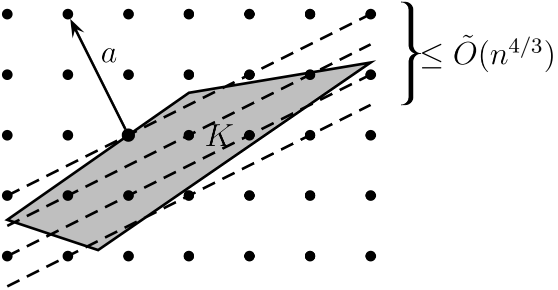

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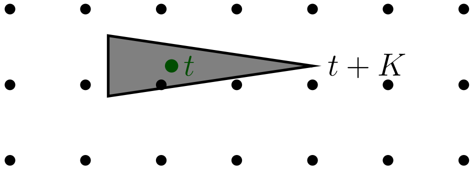
- Recurse on $\tilde{O}(n^{4/3})$ many $(n-1)$-dimensional subproblems
- $\Rightarrow \tilde{O}(n^{4/3})^n$-time algorithm

Approximate IP

Theorem (Dadush 2014)

Given convex set K with barycenter $\mathbf{0}$, can find a 2-apx to $\min\{\|x - t\|_K : x \in \mathbb{Z}^n\}$ in time $2^{O(n)}$.



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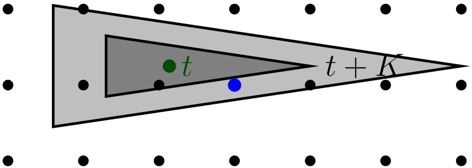

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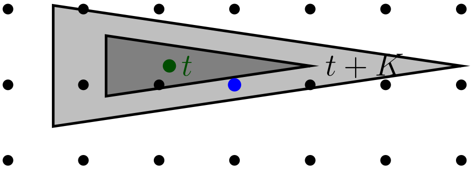

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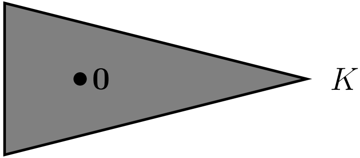

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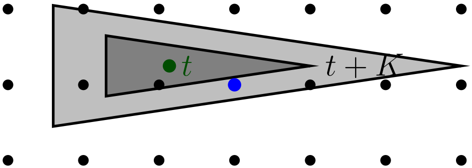

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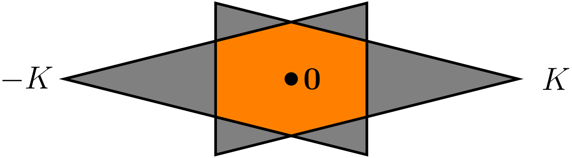

Main results

- Using approximate IP we can get:

Theorem (Dadush, Eisenbrand, R.)

There is a (simpler) $2^{O(n)}n^n$ time algorithm for integer programming. Moreover if we are given $x^* \mod 5(n+1)$ for some feasible $x^* \in K \cap \mathbb{Z}^n$ we can solve IP in time $2^{O(n)}$.

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Theorem (Dadush, Eisenbrand, R.)

The capacitated IP

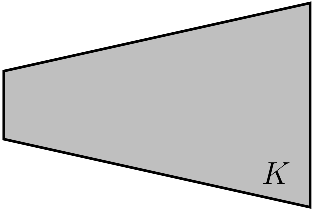
$$\max\{c^T x \mid Ax = b,\ \mathbf{0} \leq x \leq u,\ x \in \mathbb{Z}^n\}$$

can be solved in time $(\log \|u\|_\infty)^{O(n)}$.

Part I

A new $2^{O(n)}n^n$-time algorithm for general IP

Algorithm I



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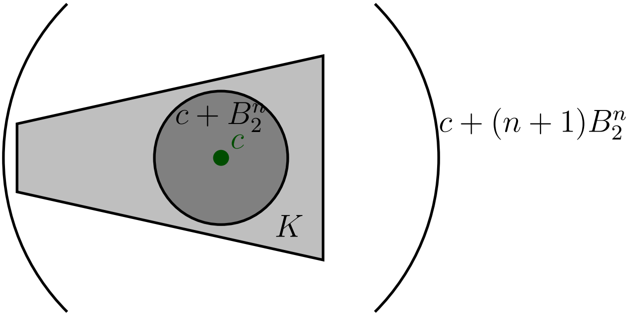

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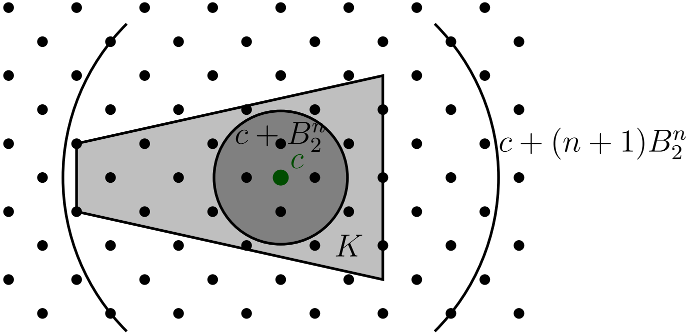

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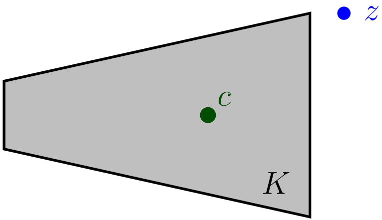

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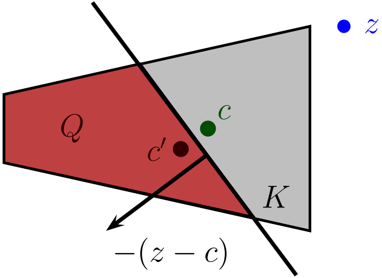

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(2) FOR $k = 1$ TO poly(n) DO

  (3) Find $y \in \Lambda$ in 2-scaling of
  $Q := K \cap \{x \mid \langle z - c, x - c\rangle \leq -\frac{1}{\text{poly}(n)}\}$

  (4) Update $z' := (1 - \frac{1}{k})z + \frac{1}{k}y$

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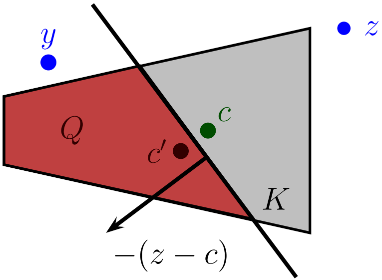

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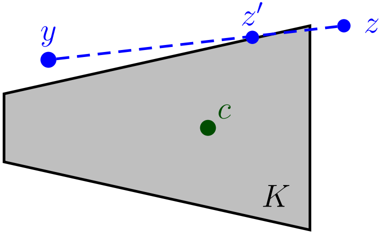

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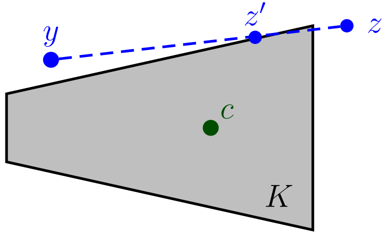

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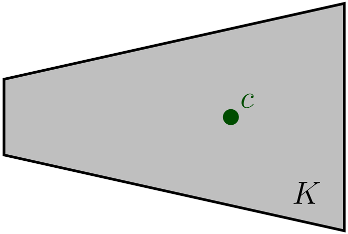
(5) IF successfull THEN find point in $K \cap \frac{\Lambda}{5(n+1)}$
(6) ELSE update $K' := K \cap \{x \mid \langle z - c, x - c\rangle \geq -\frac{1}{\text{poly}(n)}\}$

Analysis (1)

Case I: No lattice point in $Q \Rightarrow$ volume of K decreases by constant factor.



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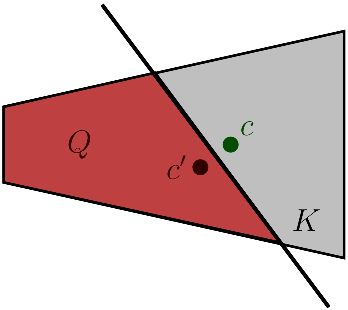

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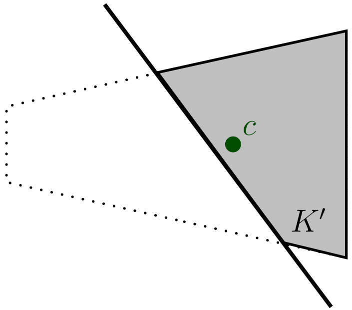

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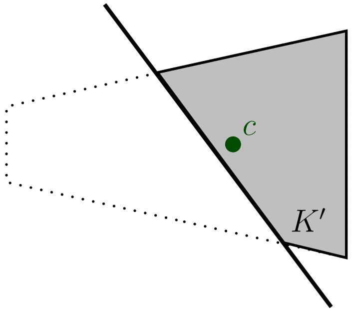
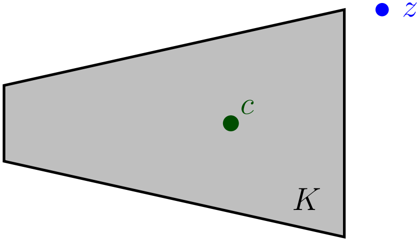


Theorem (Grünbaum)

Let $K \subseteq \mathbb{R}^n$ be a convex body. Any hyperplane H through the barycenter splits K into two parts with volume $\geq \frac{1}{e} Vol_n(K)$ each.

Anaylsis (2)

Case II: If we always find a lattice point in Q (scaled by factor 2), then c is (approx) convex combination of lattice points $X \subseteq \Lambda \cap 3(K - c) + c$.



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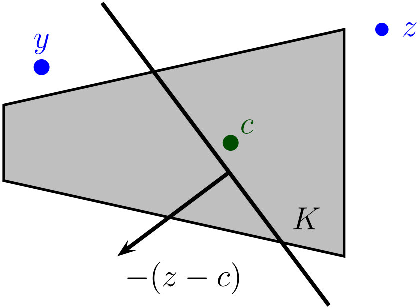

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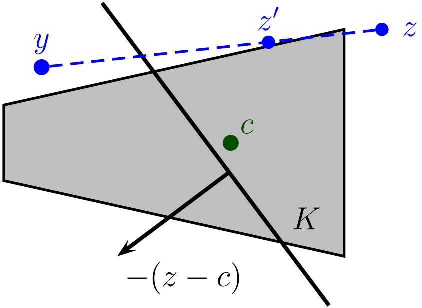
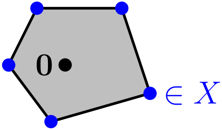


- After k iterations $\|z - c\|_2 \leq O(\frac{n}{\sqrt{k}})$.
- After $k = \text{poly}(n)$ iterations $z \approx c$

Asym. Approximate Caratheodory Thm

Lemma

If $\mathbf{0} \in P := conv(X)$, *then for any* k, *there are* $u_1, \ldots, u_k \in X$ *with* $\|\frac{1}{k}\sum_{i=1}^{k} u_i\|_P \leq \frac{n+1}{k}$.



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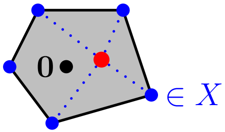

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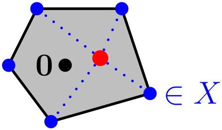
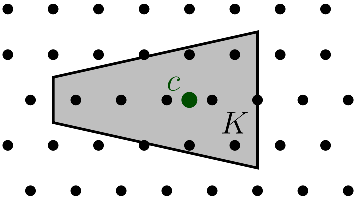


Our application: $P :=$ 3-scaling of K.

- $c \approx$ convex comb. of lattice points in 3-scaling
  $\Rightarrow$ unweighted average of $5(n+1)$ lattice points in K



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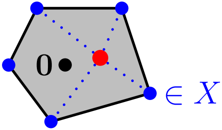

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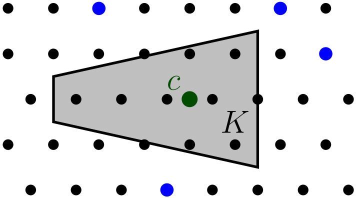

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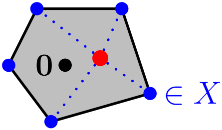

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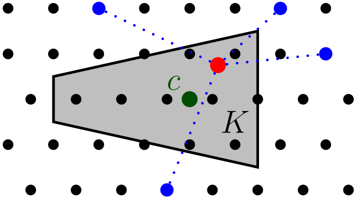

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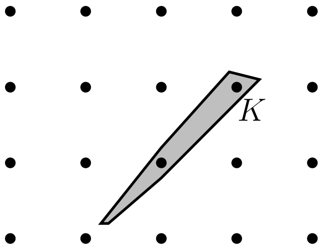

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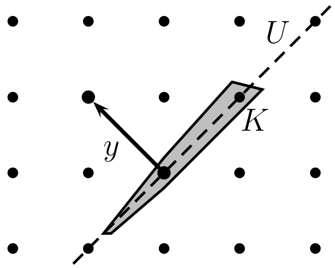

Conclusion

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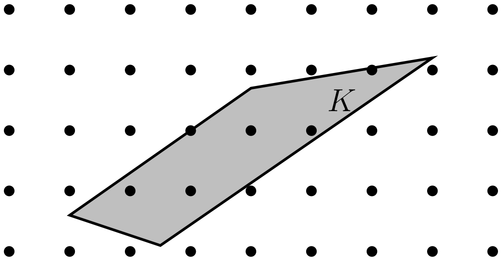

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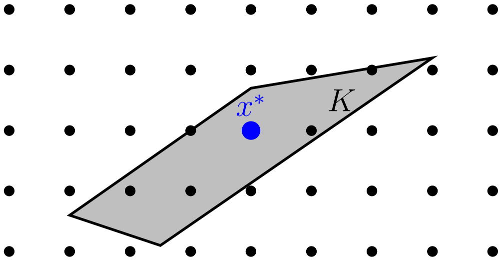

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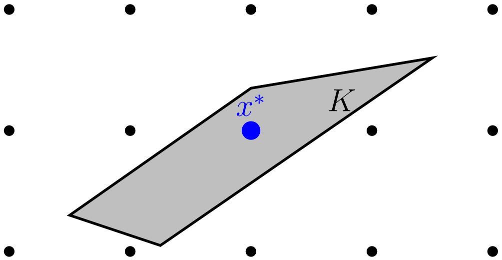

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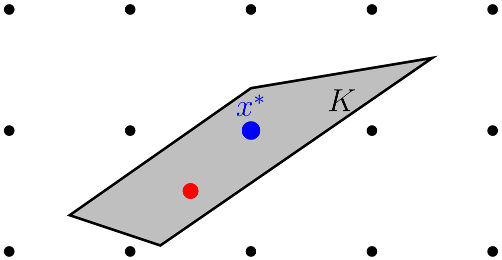

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Part II

A new algorithm for capacitated IPs

Solving IPs in equality form

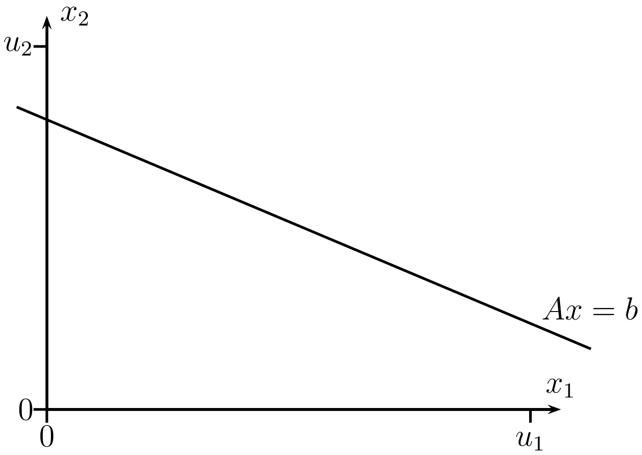
Theorem (Dadush, Eisenbrand, R.)

One can find a feasible point with

$$Ax = b,\ \mathbf{0} \leq x \leq u,\ x \in \mathbb{Z}^n$$

in time $(\log \|u\|_\infty)^{O(n)}$.

Solving IPs in equality form (2)



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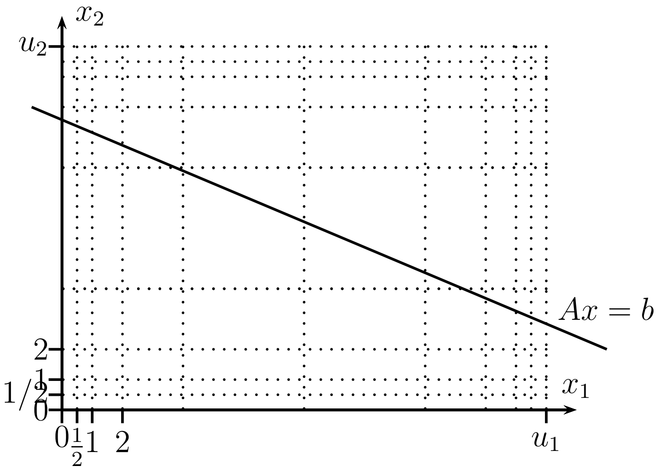

- Consider hyperplanes $\mathcal{H}$ of the form $x_i = 2^{\mathbb{Z}_{\geq -1}}$, $x_i = u_i - 2^{\mathbb{Z}_{\geq -1}}$, $x_i = 0$, $x_i = u_i$.

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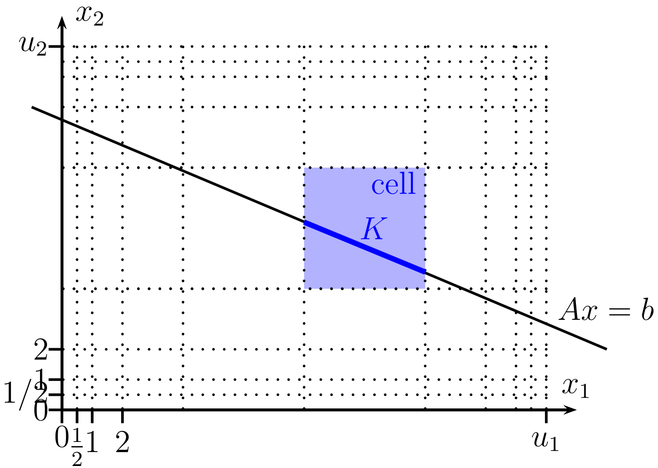

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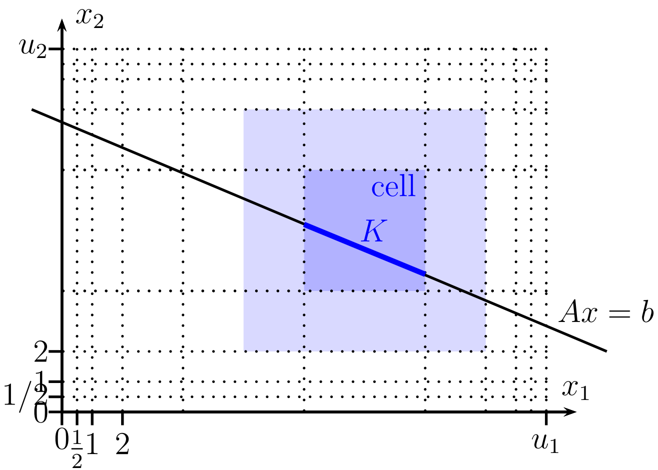

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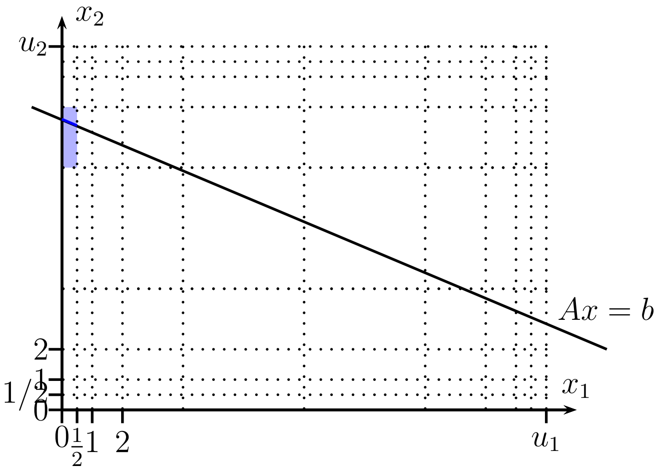

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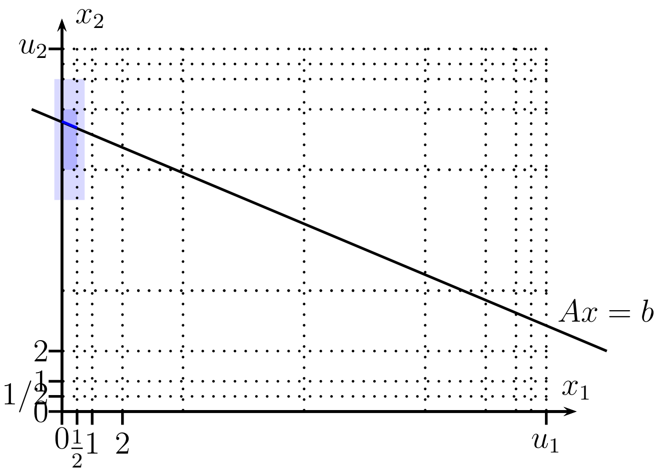

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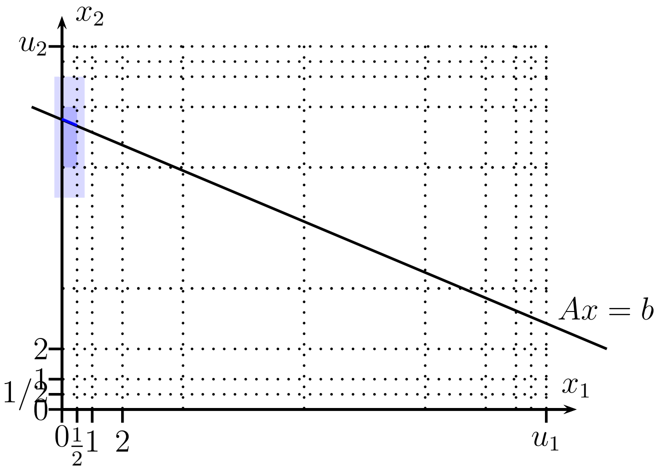

- Number of cells is $|\mathcal{H}| \leq O(\log \|u\|_\infty)^n$

Extension

Theorem (Dadush, Eisenbrand, R.)

One can solve

$$\max\{c^T x \mid Ax \le b,\ \mathbf{0} \le x \le u,\ x \in \mathbb{Z}^n\}$$

in time $n^{O(m)} \cdot (\log \|u\|_\infty)^{O(n)}$.

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- Then transform

$$A_i x \leq b_i \quad \rightarrow \quad A_i x + s_i = b_i, \quad 0 \leq s_i \leq 2^{O(n^3)} \|u\|_\infty^{O(n^2)}$$

Follow-up and open problems

More recently:

Theorem (Reis, R. '23)

One can solve any n-variable IP in time $(\log n)^{O(n)}$.

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(2) Is there a **certificate** for $K \cap \mathbb{Z}^n = \emptyset$ that can be **verified** in time $2^{O(n)}$?

(3) Can one solve **shortest vector** w.r.t. $\|\cdot\|_2$ in time $2^{O(n)}$ and **polynomial space**?

(4) Can one find a point in $K \cap \mathbb{Z}^n$ in time $2^{O(n)}$ if K is a **simplex**?

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Thanks for your attention!