On the Correlation Gap of Matroids

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Overview

1. What is Correlation Gap ($\mathcal{CG}$)?
2. $\mathcal{CG}$ of Weighted Matroid Rank
   - State of the art
   - Role of $\mathcal{CG}$ in approximation algorithms
3. Main Results
   - $\mathcal{CG}$ of weighted matroid rank $\geq$ $\mathcal{CG}$ of matroid rank
   - Improved lower bound for $\mathcal{CG}$ of matroid rank
4. Conclusion and Future Directions

Extensions of a Submodular Function

- Let $f : 2^E \to \mathbb{R}$ be a monotone submodular function.

- An extension of f is a continuous function $h : [0,1]^E \to \mathbb{R}$ such that for every $x \in [0,1]^E$,

$$h(x) = \mathbb{E}_\lambda[f(S)]$$

where λ is a probability distribution over 2^E with marginals x.

Multilinear Extension: Sample $i \in E$ with probability x_i independently

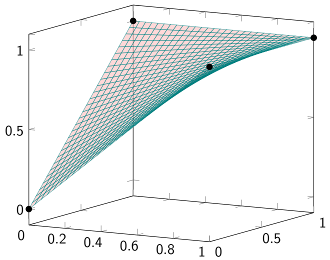
$$F(x) = \sum_{S \subseteq E} f(S) \prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i).$$

Concave Extension: Distribution which maximizes expectation

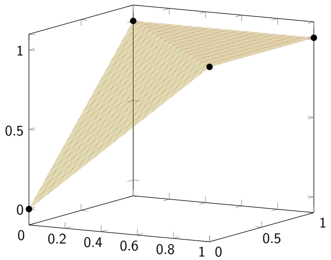
$$\hat{f}(x) = \max_{\lambda} \left\{ \sum_{S \subseteq E} \lambda_S f(S) : \sum_{S \ni i} \lambda_S = x_i \ \forall i \in E, \sum_{S \subseteq E} \lambda_S = 1, \lambda \geq 0 \right\}.$$

Example

- Let f be the rank function of a rank-1 uniform matroid on 2 elements.



$F(x)$



$\hat{f}(x)$

- Monotone.
- Concave along any e_i.
- Convex along any $e_i - e_j$.

- Monotone.
- Concave.
- Piecewise affine.

Correlation Gap

Def: [Agrawal et al. '12] The correlation gap of f is the ratio

$$\mathcal{CG}(f) = \min_{x \in [0,1]^E} \frac{F(x)}{\hat{f}(x)}.$$

- [Calinescu et al. '07] For any monotone submodular function f,

$$\mathcal{CG}(f) \geq 1 - \frac{1}{e}.$$

- Tight for the rank function of a rank-1 uniform matroid. Letting $n = |E|$,

$$\frac{F\left(\frac{1}{n}\mathbb{1}\right)}{\hat{f}\left(\frac{1}{n}\mathbb{1}\right)} = \frac{1 - \left(1 - \frac{1}{n}\right)^n}{n \cdot \frac{1}{n}} \to 1 - \frac{1}{e} \qquad \text{as } n \to \infty.$$

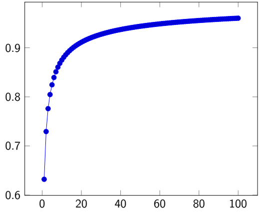
$\mathcal{CG}$ of Weighted Matroid Rank

Def: Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and weights $w \in \mathbb{R}_+^E$, the weighted matroid rank function is

$$r_w(S) := \max\{w(T) : T \subseteq S, T \in \mathcal{I}\}.$$

- r_w is monotone submodular $\implies \mathcal{CG}(r_w) \geq 1 - e^{-1}$.
- [Yan '11, Barman et al. '20] For a rank-ℓ uniform matroid, its (unweighted) rank function r has correlation gap

$$\mathcal{CG}(r) = 1 - \frac{e^{-\ell}\ell^{\ell}}{\ell!}.$$



$\mathcal{CG}$ is Approximation Guarantee

Problem: [Calinescu et al. '07] Given a sum of weighted matroid rank functions $f = \sum_{i=1}^{m} f_i$ and a matroid $\mathcal{M} = (E, \mathcal{I})$, solve

$$\max_{S \in \mathcal{I}} f(S).$$

- NP-hard as it captures the maximum coverage problem:

MaxCover: Given a family of sets $E \subseteq 2^N$, select k sets to cover as many elements in N as possible.

Reduction: Let $\mathcal{M} = (E, \mathcal{I})$ be a uniform matroid of rank k. For each $i \in N$, define $f_i : 2^E \to \mathbb{Z}_+$ as

$$f_i(S) = \begin{cases} 1, & \text{if } \exists \text{ a set in } S \text{ containing } i, \\ 0, & \text{otherwise.} \end{cases}$$

- Each f_i is the weighted rank function of a rank-1 uniform matroid.

$\mathcal{CG}$ is Approximation Guarantee

- A generalization of **MaxCover** is the maximum multicoverage problem.

Max-ℓ-Cover: Given a family of sets $E \subseteq 2^N$, select k sets to cover as many elements in N as possible, where each element is counted up to ℓ times.

Reduction: Let $\mathcal{M} = (E, \mathcal{I})$ be a uniform matroid of rank k. For each $i \in N$, define $f_i : 2^E \to \mathbb{Z}_+$ as

$$f_i(S) = \min(\# \text{ of sets in } S \text{ containing } i, \ell).$$

- Each f_i is the weighted rank function of a rank-ℓ uniform matroid.
- [Barman et al. '20] gave a tight $1 - \frac{e^{-\ell}\ell^\ell}{\ell!}$ approximation algorithm.
- Applications in list decoding and approval voting.

$\mathcal{CG}$ is Approximation Guarantee

Algorithm: [Calinescu et al. '07, Shioura '09]

1. Let x^* be an optimal solution to the following LP

$$\max \left\{ \sum_{i=1}^{m} \hat{f}_i(x) : x \in \text{independent set polytope of } \mathcal{M} \right\}.$$

2. Round x^* to $x' \in \{0,1\}^E$ using pipage rounding.

Analysis:

$$f(x') = F(x') \geq F(x^*) = \sum_{i=1}^{m} F_i(x^*) \geq \sum_{i=1}^{m} \mathcal{CG}(f_i)\, \hat{f}_i(x^*) \geq \min_{i \in [m]} \mathcal{CG}(f_i) \cdot \mathrm{OPT}.$$

- $\mathcal{CG}$ is also the **approximation guarantee** in sequential posted price mechanism and contention resolution schemes.

Main Results

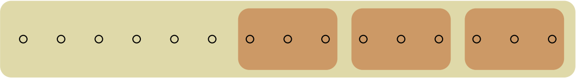
Theorem 1

For any matroid $\mathcal{M} = (E, \mathcal{I})$ with rank function r,

$$\inf_{w \in \mathbb{R}_+^E} \mathcal{CG}(r_w) = \mathcal{CG}(r).$$

Goal: Identify matroid parameters which govern $\mathcal{CG}(r)$.

- Parameterization by rank ρ or girth γ only yields $\mathcal{CG}(r) = 1 - 1/e$.
  - Partition matroid with ρ rank-1 parts.
  - Union of a rank-$(\gamma - 1)$ uniform matroid and **many** rank-1 uniform matroids.

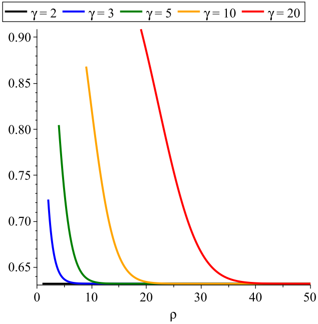


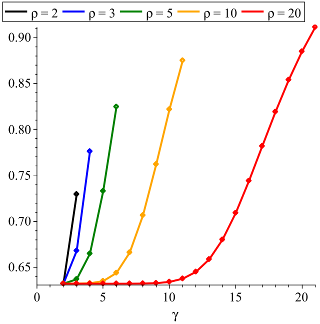
Main Results

Theorem 2

For any matroid $\mathcal{M}$ with rank function r, rank ρ, and girth $\gamma > 1$,

$$\mathcal{CG}(r) \geq 1 - \frac{1}{e} + \frac{e^{-\rho}}{\rho} \left(\sum_{i=0}^{\gamma-2} (\gamma - 1 - i) \left[\binom{\rho}{i} (e-1)^i - \frac{\rho^i}{i!} \right] \right) \geq 1 - \frac{1}{e}\,.$$





Proof Ideas

Theorem 1:

- Let $w \in \mathbb{R}_+^E$ where $\mathcal{CG}(r_w) < \mathcal{CG}(r)$ with the fewest distinct values.
- Can increase weights in a value class without increasing $\mathcal{CG}(r_w)$. ↯

Theorem 2:

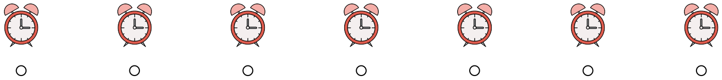
- Let r be the rank function of a matroid $\mathcal{M}$ with rank ρ and girth γ.
- $\exists$ a point x in the independent set polytope of $\mathcal{M}$ such that

$$\mathcal{CG}(r) = \frac{R(x)}{\hat{r}(x)} = \frac{R(x)}{\mathbb{1}^\top x}$$

- To bound $R(x)$, decompose $r = g + h$ such that g is the rank function of a rank-ℓ uniform matroid, where $\ell := \gamma - 1$.
  - Bounding $G(x)$: simple application of [Yan '11, Barman et al. '20]
  - Bounding $H(x)$: Poisson clock analysis [Calinescu et al. '07]

Poisson Clock Analysis

- Construct a random set $Q(1)$ as follows:
  - Put a Poisson clock of rate x_i on each element i (independent).
  - From time $t = 0$ to 1, add i to $Q(t)$ whenever its clock rings.



- By monotonicity of h, $H(x) \geq H(1 - e^{-x}) = \mathbb{E}[h(Q(1))]$.

- In [Calinescu et al. '07], $\frac{d}{dt}\mathbb{E}[h(Q(t))] \geq \hat{h}(x) - \mathbb{E}[h(Q(t))]$ for all t.

$$\Rightarrow \ \mathbb{E}[h(Q(1))] \geq (1 - e^{-1})\hat{h}(x).$$

- For us, $\frac{d}{dt}\mathbb{E}[h(Q(t))] \geq \hat{r}(x) - \ell - \mathbb{E}[h(Q(t))]$ only if $|Q(t)| \geq \ell$.

$$\Rightarrow \ \mathbb{E}[h(Q(1)) \,|\, |Q(t)| \geq \ell \ \forall t \geq T'] \geq (1 - e^{T'-1})(\hat{r}(x) - \ell).$$

Conclusion

Summary of Results:

1. $\mathcal{CG}$ of weighted matroid rank is minimized under uniform weights.
2. Improved lower bound on $\mathcal{CG}(r)$ parameterized by rank and girth.
   - When girth is fixed, $\mathcal{CG} \downarrow$ as rank $\uparrow$.
   - When rank is fixed, $\mathcal{CG} \uparrow$ as girth $\uparrow$.

Future Directions:

1. Upper and lower bounds do not match. There exists a rank-ρ girth-γ matroid with

$$\mathcal{CG}(r) \leq 1 - \frac{1}{e} + \frac{\gamma - 1}{e\rho}.$$

2. Find better matroid parameters that govern $\mathcal{CG}(r)$.