Cut-Sufficient Directed 2-Commodity Multiflow Topologies

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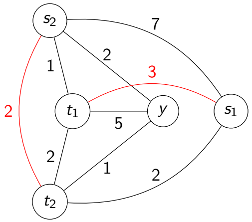
Multicommodity Flow Problem

Topology: supply-demand graph pair (G, H)

- $E(H) = \{s_1 t_1, \ldots, s_k t_k\}$ (H is always drawn red)

Weights: capacities $u \in \mathbb{Z}_{\geq 0}^{E(G)}$, demand weights $d \in \mathbb{Z}_{\geq 0}^{E(H)}$



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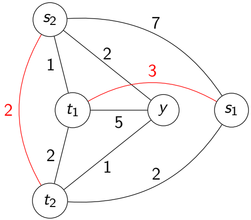
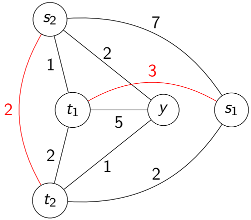
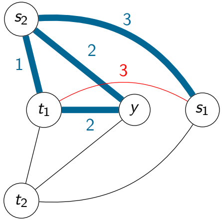


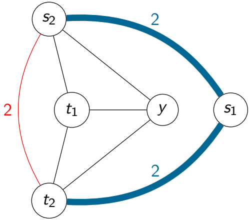
Feasible multiflow: family of $s_i t_i$-flows $f^{s_1 t_1}, \ldots f^{s_k t_k}$ that

- satisfies demands: $f^{s_i t_i}$ has size $d(s_i t_i)$
- sum respects capacities: $\sum_i f^{s_i t_i}(e) \leq u(e)$

Multicommodity Flow Example







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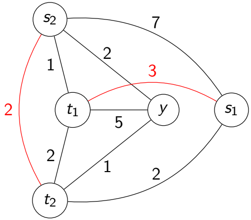
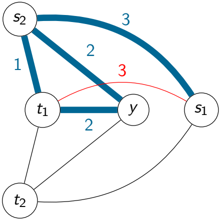
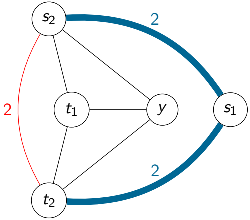






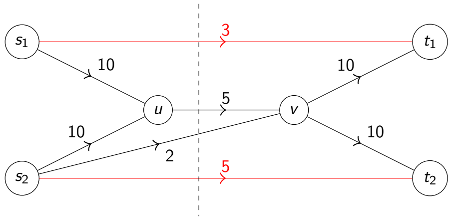
- Satisfy demands? ✓
- Sum respects capacities? ✓

The Cut Condition

Definition (Cut Condition)

The *cut condition* holds if, for all $S \subseteq V(G)$:

- (undirected) $u\left(\delta_G(S)\right) \geq d\left(\delta_G(S)\right)$
- (directed) $u\left(\delta^+_G(S)\right) \geq d\left(\delta^+_G(S)\right)$



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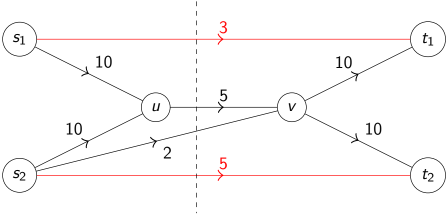

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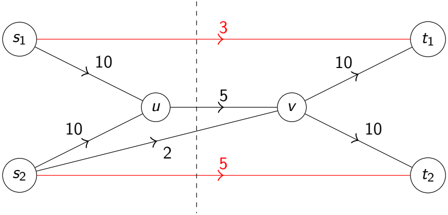


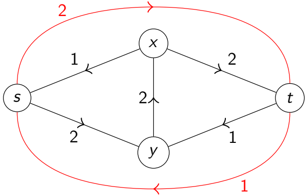
The cut condition is necessary for feasibility.

Theorem (Max-Flow Min-Cut, 1956)

If $|E(H)| = 1$ or H has a single source/single sink, the cut condition is also sufficient.

The Cut Condition is Not Enough

The cut condition is not always sufficient.



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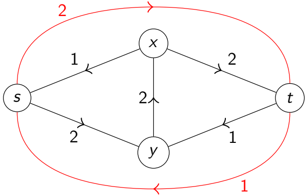

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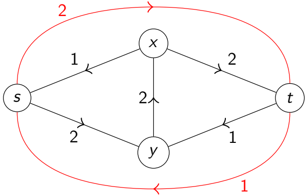

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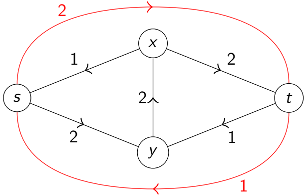

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Here $\alpha^* > 1$, despite the cut condition being satisfied.

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For a given topology (G, H), we say weights (u, d) are *bad* when:

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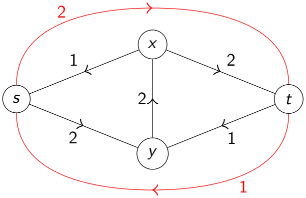

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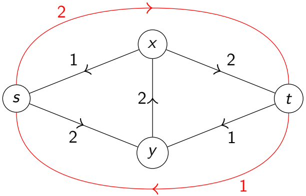

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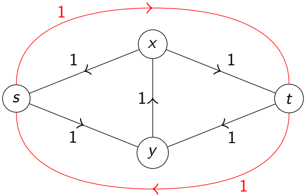

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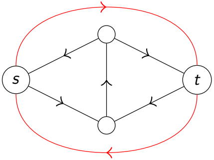

From before: $\text{gap}(G, H) \geq 2$ for this topology.

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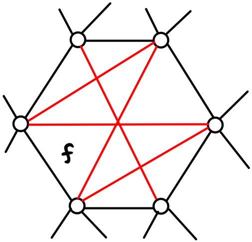
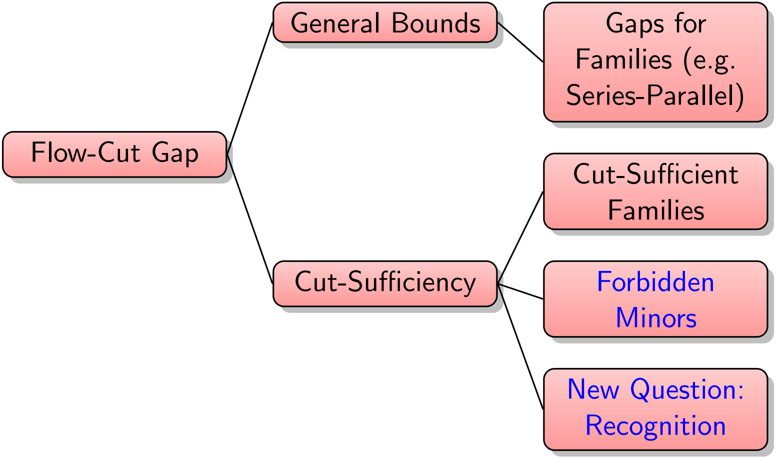
- *(undirected) G is planar and $V(H) \subseteq f$ for some face f (Okamura and Seymour, 1981).*



Gap Questions



Our focus: directed topologies (less well understood)

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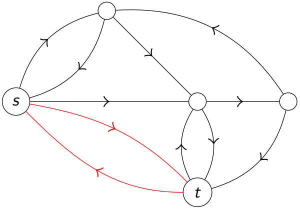

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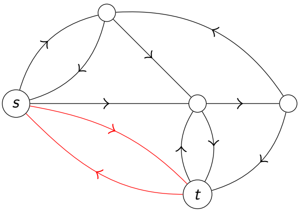

Later...

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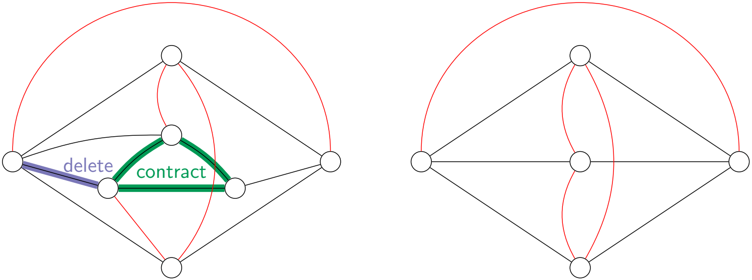

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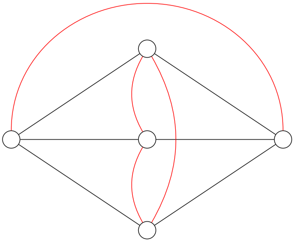

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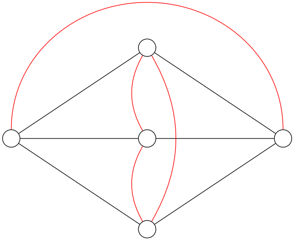

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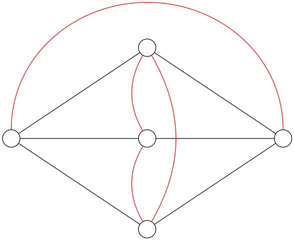

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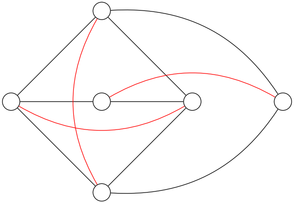

The bad-K_4-pair.

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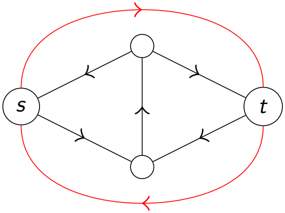

The bad dual triangles.

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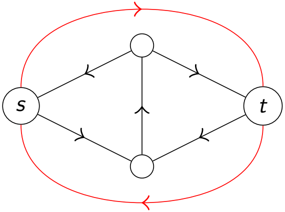

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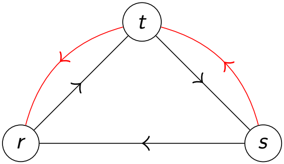

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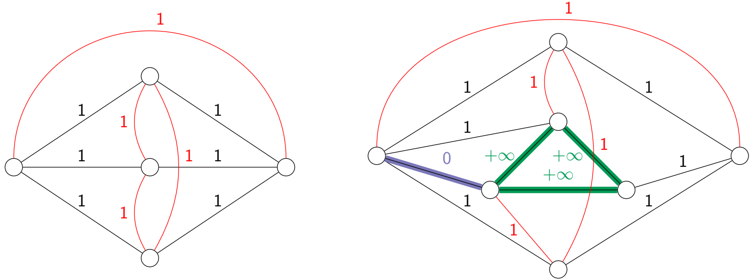

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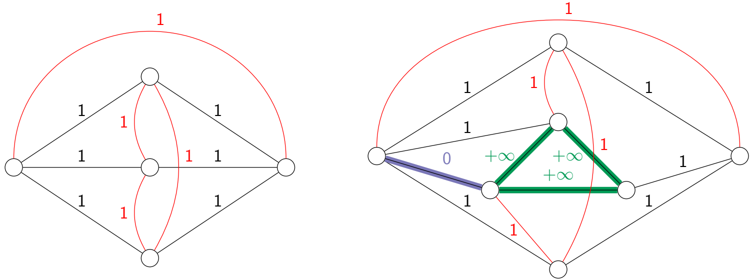

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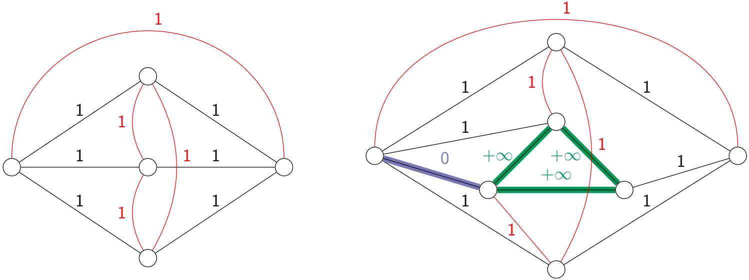

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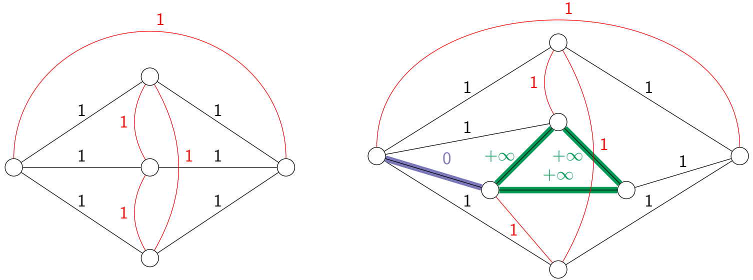

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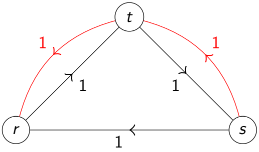

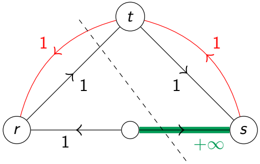

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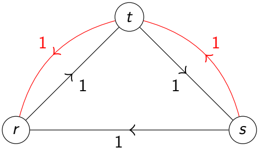

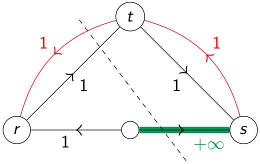

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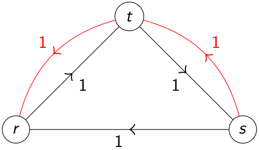

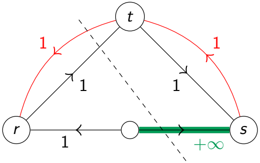

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Directed Relevant Minors

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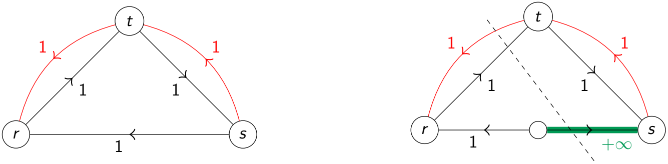

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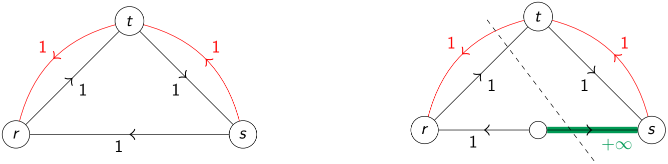

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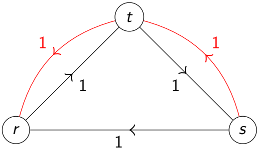

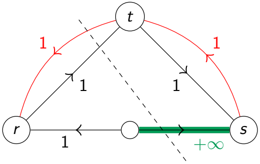

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$\mathcal{CS}$ is relevant-minor-closed.

Useful Operations

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The following operations produce relevant minors:

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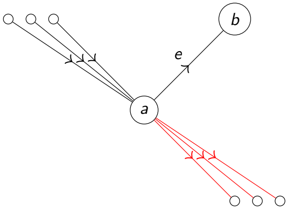

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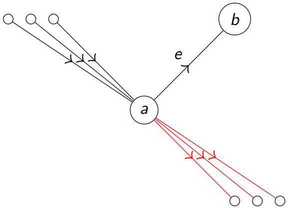

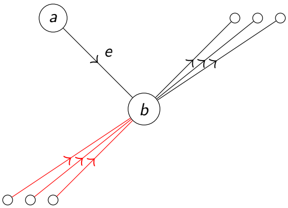

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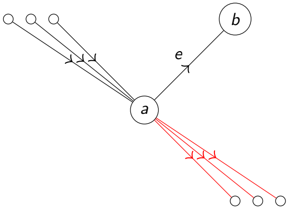

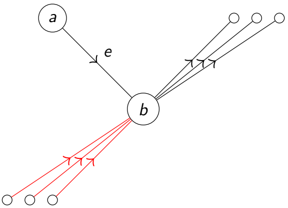

Importantly: cycles and subdivisions.

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Special cases of general property: no "new $s_i t_j$-connectivity".

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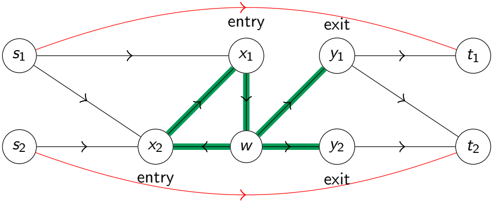

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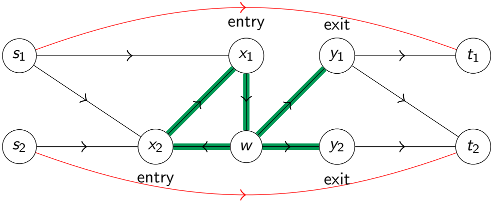

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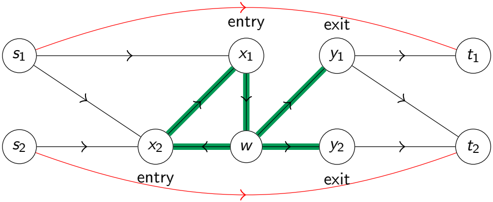

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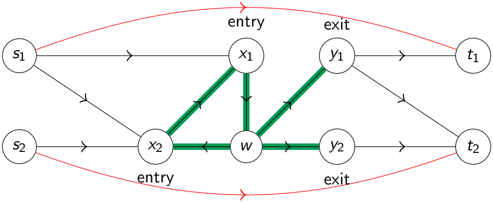

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Contractions of EEC sets produce relevant minors.

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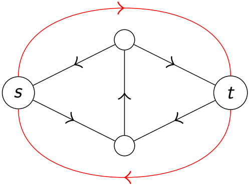

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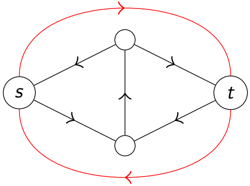

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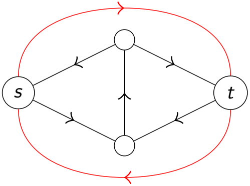

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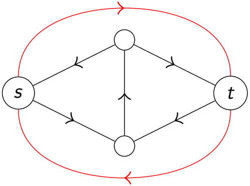

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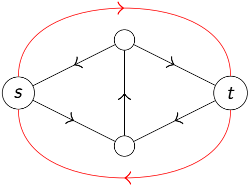

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- Output: either
  - feasible multiflow, OR
  - show that (G, H) has the bad dual triangles as a relevant minor.

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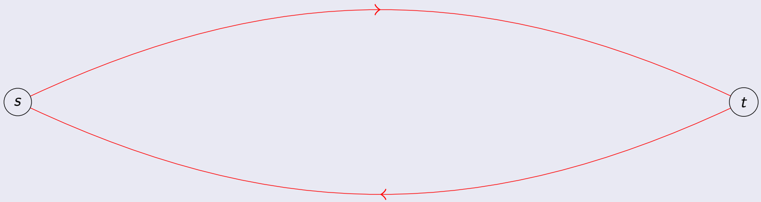

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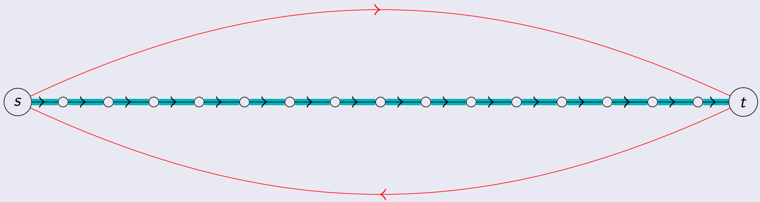

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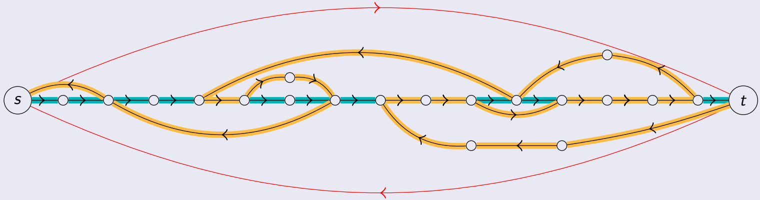

Simplifying Path Interactions

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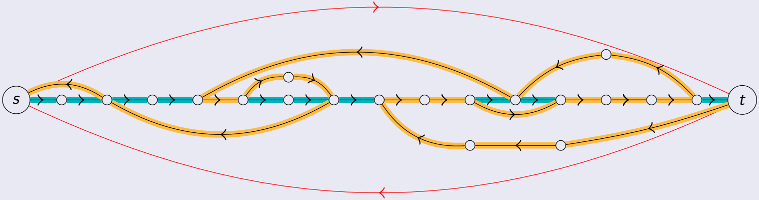

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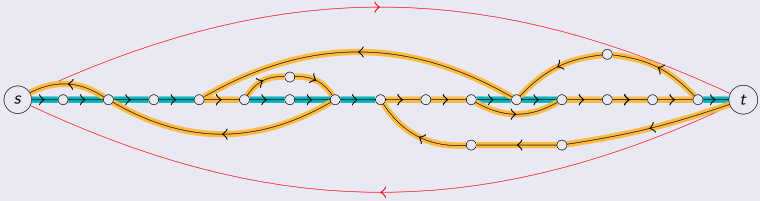

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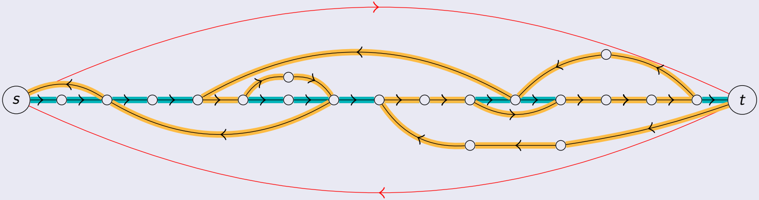

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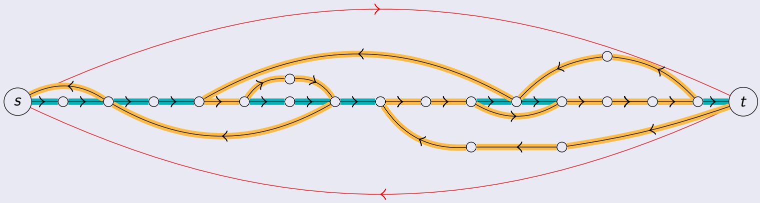

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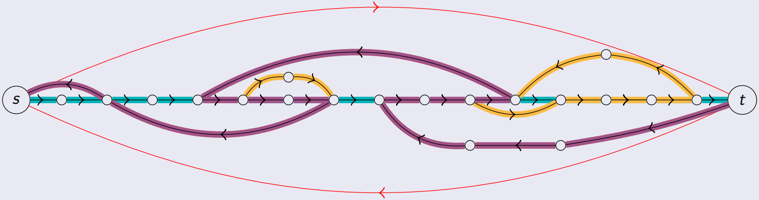

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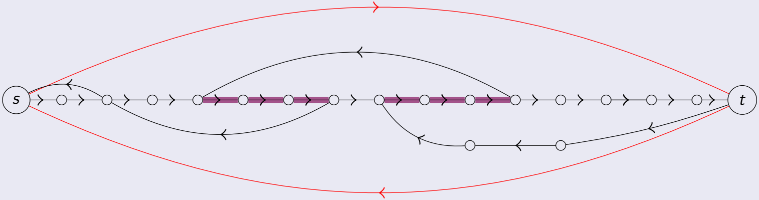

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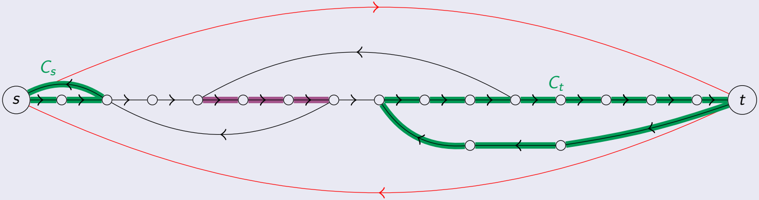

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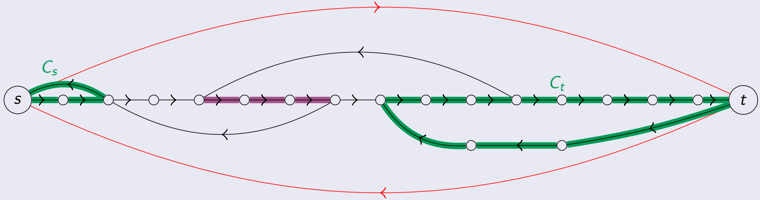
- Be careful, want to preserve a shared edge



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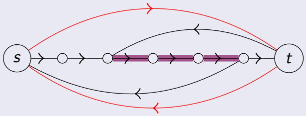

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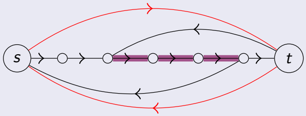

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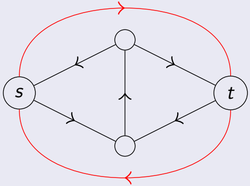

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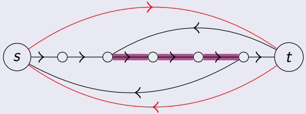

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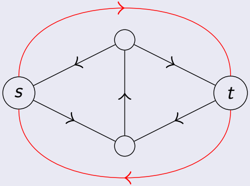

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Similar pf. for 2-path demands (more cases, things to contract).

Toward NP-hardness

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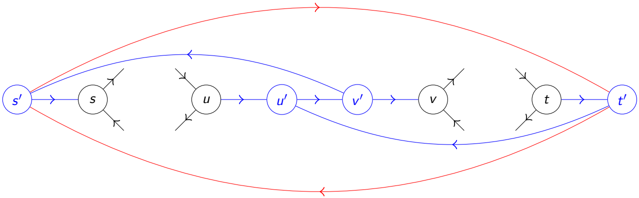

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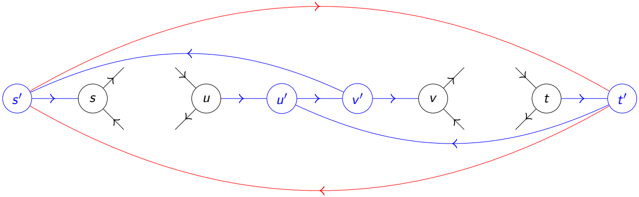

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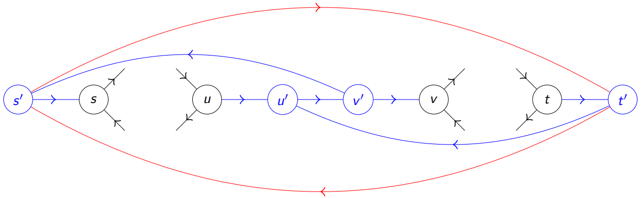

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NP-hardness Reduction

Theorem

It is NP-hard to recognize if directed (G, H) is cut-sufficient, even if H is fixed as a 2-cycle (roundtrip demands).

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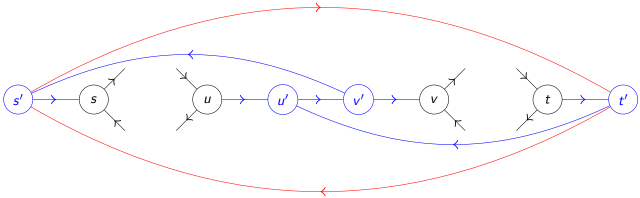

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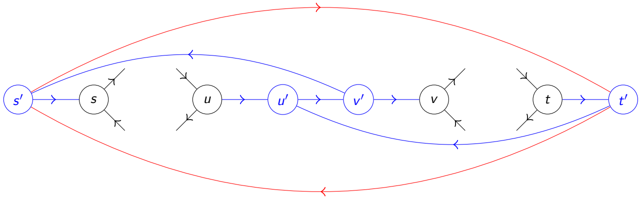

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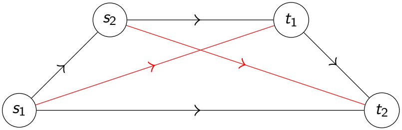
Towards a Complete 2-Commodity Characterization

The only remaining case (aside from single-source/sink) is *2-matching* demands.



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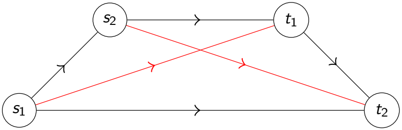
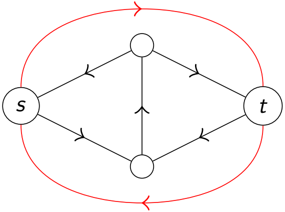
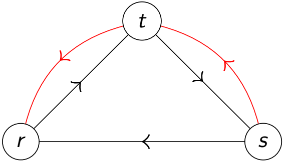


Conjecture

If (G, H) has 2-matching demands, then it is cut-sufficient if and only if it does not contain the bad triangle or bad dual triangles as a relevant minor.





Partial Result

Approach from earlier...

Algorithm

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Proposition

If (G, H) has 2-matching demands and bad weights (u, d) where $d(s_i t_i) = 1$ for $i = 1, 2$, then it contains the bad triangle or bad dual triangles as a relevant minor.