Problem Set for IPCO 2023 Summer School

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1. We proved in the lecture the upper bound $O(n2^n \log(R))$ on information complexity of *pure integer feasibility*.

   (a) Prove the upper bound $O\left(d(n+d)2^n \log\left(\frac{R}{\rho}\right)\right)$ for testing mixed-integer feasibility. Recall that we are considering the following family of closed, convex sets C as our constraints: $C \subseteq [-R, R]^{n+d}$ and if $C \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ is nonempty, then there exists $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathbb{Z}^n \times \mathbb{R}^d$ such that $\{(\bar{\mathbf{x}}, \mathbf{y}) : \|\mathbf{y} - \bar{\mathbf{y}}\|_\infty \leq \rho)\} \subseteq C$.

   (b) Prove the upper bound $O\left(d(n+d)2^n \log\left(\frac{MR}{\rho\varepsilon}\right)\right)$ for finding ε-approximate solutions to the problem $\min\{f(\mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in C, \mathbf{x} \in \mathbb{Z}^n, \mathbf{y} \in \mathbb{R}^d)\}$. Recall that we assume the function f is M-Lipschitz, i.e., $|f(\mathbf{z}) - f(\mathbf{z}')| \leq M\|\mathbf{z} - \mathbf{z}'\|_\infty$.

   Hint: Generalize the centerpoint algorithm discussed in the lecture that used the fact that for any probability distribution μ supported on $\mathbb{Z}^n \times \mathbb{R}^d$, there always exists $\mathbf{z}^\star \in \mathbb{Z}^n \times \mathbb{R}^d$ such that every halfspace H containing $\mathbf{z}^\star$ satisfies $\mu(H) \geq \frac{1}{2^n(d+1)}$.

   Moreover, for the general optimization upper bound, the following lemma could be useful.

   LEMMA 0.1. *Let $C \subseteq \mathbb{R}^k$ be a closed, convex set such that $\{\mathbf{z} \in \mathbb{R}^k : \|\mathbf{z} - \mathbf{a}\|_\infty \leq \rho\} \subseteq C \subseteq [-R, R]^k$, for some $R, \rho \in \mathbb{R}_+$ and $\mathbf{a} \in \mathbb{R}^k$. Let $f : \mathbb{R}^k \to \mathbb{R}$ be a convex function that is M-Lipschitz continuous. For any $\varepsilon \leq 2MR$ and for any $\mathbf{z}^\star \in C$, the set $\{\mathbf{z} \in C : f(\mathbf{z}) \leq f(\mathbf{z}^\star) + \varepsilon\}$ contains an $\|\cdot\|_\infty$ ball of radius $\frac{\varepsilon\rho}{2MR}$ with center lying on the line segment between $\mathbf{z}^\star$ and $\mathbf{a}$.*

   Prove this lemma.

2. Prove the "continuous" version of the centerpoint guarantee, using Helly's theorem. Recall Helly's theorem says

   > Given convex sets $C_1, \ldots, C_k \subseteq \mathbb{R}^d$, if $C_1 \cap \ldots \cap C_k = \emptyset$, then there exist $i_1, \ldots, i_j$ with $j \leq d+1$ such that $C_{i_1} \cap \ldots C_{i_j} = \emptyset$.

   Show this implies that for any probability distribution μ on $\mathbb{R}^d$, there always exists $\mathbf{z}^\star \in \mathbb{R}^d$ such that every halfspace H containing $\mathbf{z}^\star$ satisfies $\mu(H) \geq \frac{1}{d+1}$. Use this idea to prove the general mixed-integer version stated in Question 1 above.

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