1. We proved in the lecture the upper bound $O(n^2 \log(R))$ on information complexity of pure integer feasibility.

(a) Prove the upper bound $O\left(d(n + d)2^n \log\left(\frac{R}{\rho}\right)\right)$ for testing mixed-integer feasibility. Recall that we are considering the following family of closed, convex sets $C$ as our constraints: $C \subseteq [-R, R]^{n+d}$ and if $C \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ is nonempty, then there exists $(\bar{x}, \bar{y}) \in \mathbb{Z}^n \times \mathbb{R}^d$ such that $\{(\bar{x}, y) : \|y - \bar{y}\|_\infty \leq \rho\} \subseteq C$.

(b) Prove the upper bound $O\left(d(n + d)2^n \log\left(\frac{MR}{\rho}\right)\right)$ for finding $\varepsilon$-approximate solutions to the problem $\min \{f(x, y) : (x, y) \in C, x \in \mathbb{Z}^n, y \in \mathbb{R}^d\}$. Recall that we assume the function $f$ is $M$-Lipschitz, i.e., $|f(z) - f(z')| \leq M\|z - z'\|_\infty$.

Hint: Generalize the centerpoint algorithm discussed in the lecture that used the fact that for any probability distribution $\mu$ supported on $\mathbb{Z}^n \times \mathbb{R}^d$, there always exists $z^* \in \mathbb{Z}^n \times \mathbb{R}^d$ such that every halfspace $H$ containing $z^*$ satisfies $\mu(H) \geq \frac{1}{2^{(d+1)}}$.

Moreover, for the general optimization upper bound, the following lemma could be useful.

**Lemma 0.1.** Let $C \subseteq \mathbb{R}^k$ be a closed, convex set such that $\{z \in \mathbb{R}^k : \|z - a\|_\infty \leq \rho\} \subseteq C \subseteq [-R, R]^k$, for some $R, \rho \in \mathbb{R}_+$ and $a \in \mathbb{R}^k$. Let $f : \mathbb{R}^k \rightarrow \mathbb{R}$ be a convex function that is $M$-Lipschitz continuous. For any $\varepsilon \leq 2MR$ and for any $z^* \in C$, the set $\{z \in C : f(z) \leq f(z^*) + \varepsilon\}$ contains an $\|\cdot\|_\infty$ ball of radius $\frac{\rho}{2MR}$ with center lying on the line segment between $z^*$ and $a$.

Prove this lemma.

2. Prove the "continuous" version of the centerpoint guarantee, using Helly's theorem. Recall Helly's theorem says

Given convex sets $C_1, \ldots, C_k \subseteq \mathbb{R}^d$, if $C_1 \cap \ldots \cap C_k = \emptyset$, then there exist $i_1, \ldots, i_j$ with $j \leq d + 1$ such that $C_{i_1} \cap \ldots \cap C_{i_j} = \emptyset$.

Show this implies that for any probability distribution $\mu$ on $\mathbb{R}^d$, there always exists $z^* \in \mathbb{R}^d$ such that every halfspace $H$ containing $z^*$ satisfies $\mu(H) \geq \frac{1}{2^{d+1}}$. Use this idea to prove the general mixed-integer version stated in Question 1 above.