ReLU Neural Networks of Polynomial Size for Exact Maximum Flow Computation

Christoph Hertrich

LSE

Leon Sering

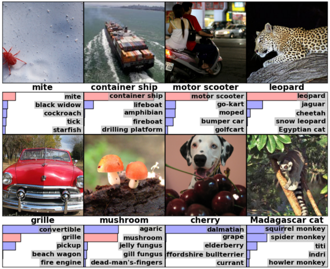
ETH zürich

started at:

TU berlin

IPCO, Madison, WI, USA
June 21, 2023

Neural Networks in Action



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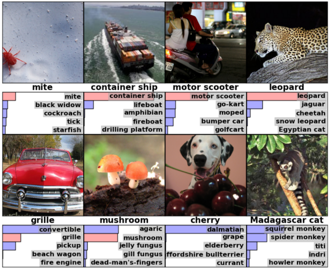

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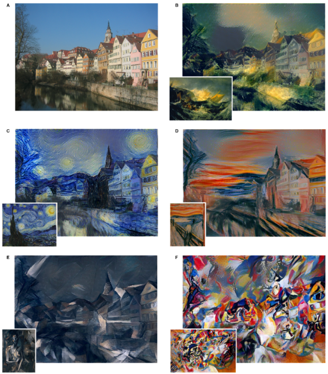

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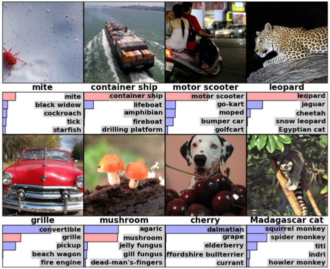

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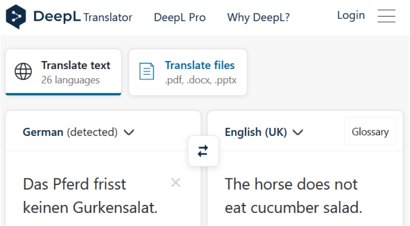
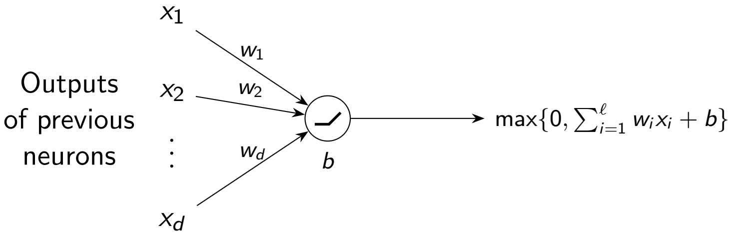


Screenshot `deepl.com` (Feb 18, 2022)

A Single ReLU Neuron



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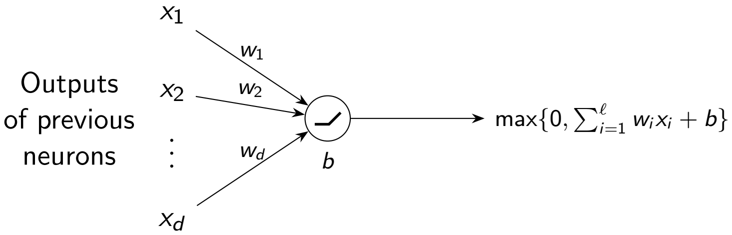
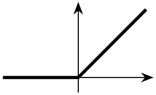


Rectified linear unit (ReLU): $\mathrm{relu}(x) = \max\{0, x\}$



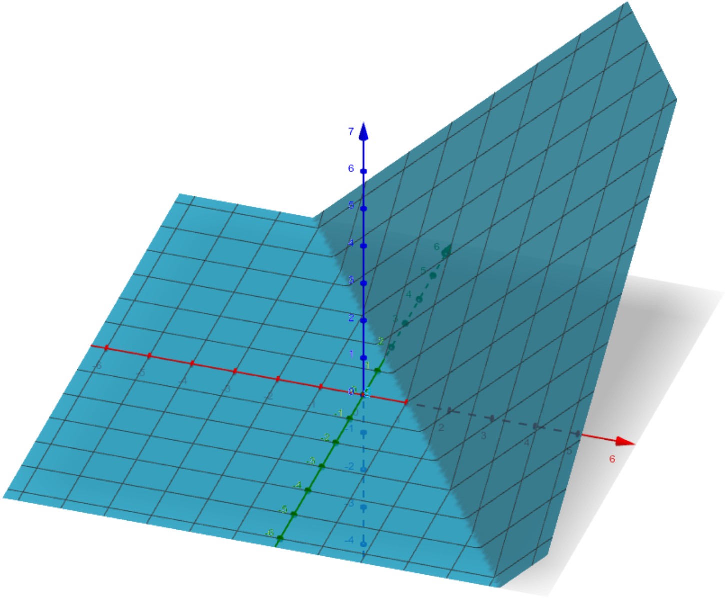
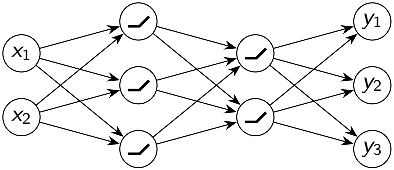


Image created with GeoGebra (geogebra.org)

ReLU Feedforward Neural Networks

- Acyclic (layered) digraph of ReLU neurons



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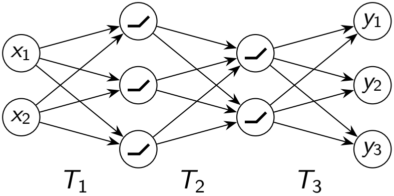

- Computes function

$$T_k \circ \text{relu} \circ T_{k-1} \circ \cdots \circ T_2 \circ \text{relu} \circ T_1$$

with affine transformations T_i.

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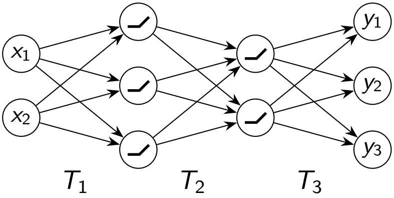

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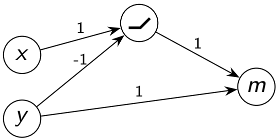
- Example: depth 3, size 5.

Example: Computing the Maximum of Two Numbers

$$\max\{x, y\} = \max\{x - y, 0\} + y$$



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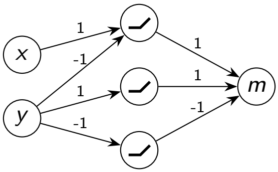
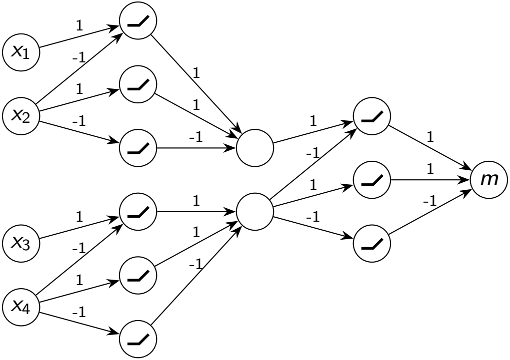


Example: Computing the Maximum of Four Numbers



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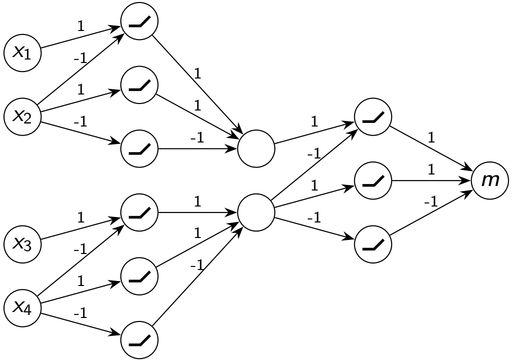

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  Max of n numbers with depth $\mathcal{O}(\log n)$ and size $\mathcal{O}(n)$.

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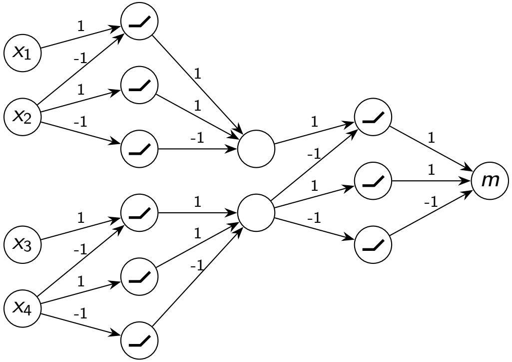

- Inductively:
  Max of n numbers with depth $\mathcal{O}(\log n)$ and size $\mathcal{O}(n)$.
- Minimum similarly.

Representing Arbitrary Piecewise Linear Functions

Observation

Every function represented by a ReLU NN is continuous and piecewise linear (CPWL).

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Theorem (Arora, Basu, Mianjy, Mukherjee (ICLR 2018))

Every CPWL function $f : \mathbb{R}^n \to \mathbb{R}$ can be represented by a ReLU NN with depth $\mathcal{O}(\log n)$.

Open Questions

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1. Is logarithmic depth best possible?

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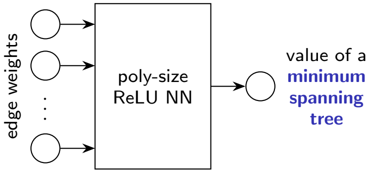
Every CPWL function $f : \mathbb{R}^n \to \mathbb{R}$ can be represented by a ReLU NN with depth $\mathcal{O}(\log n)$.

1. Is logarithmic depth best possible?

2. **Which functions can we represent with polynomial size?**

Our Results

Hertrich, Sering (IPCO 2023)



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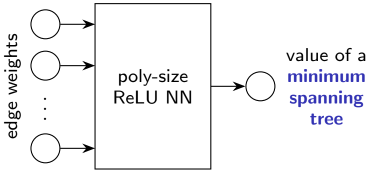

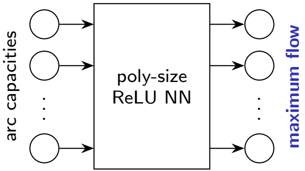

Neural Networks as Model of **Real-Valued** Computation

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Neural Networks

$=$

Arithmetic circuits with (weighted) sums and maxima gates

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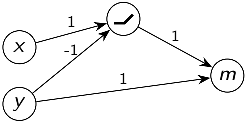
Polynomial-Size NNs

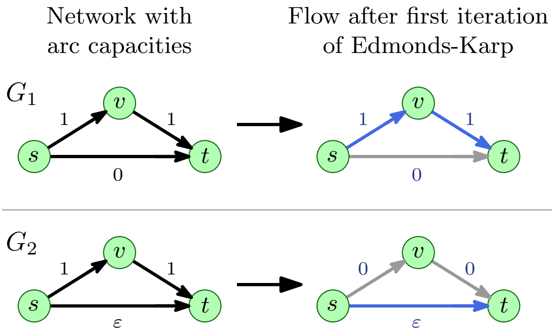


$\Leftrightarrow$

Strongly polynomial algorithms
with restricted set of operations

Input: $x, y \in \mathbb{R}$
$v_1 = x - y$
$v_2 = \max\{0, v_1\}$
$m = v_2 + y$
return m.

Why Edmonds-Karp does not work



What to do instead?

Assume s-t-distance of length $\geq k$ in residual network. Need to find an augmenting s-t-flow which

- uses only arcs along paths of length k,
- is feasible in the residual network, and
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Then: Similar analysis to Edmonds-Karp

How to find augmenting flow

1. Compute fattest path values $a_{i,v}$: maximum amount of flow that can be pushed on a single path of length exactly i from v to t:

$$a_{i,v} = \max_{vw \in E} \{\min\{c_{vw}, a_{i-1,w}\}\}$$

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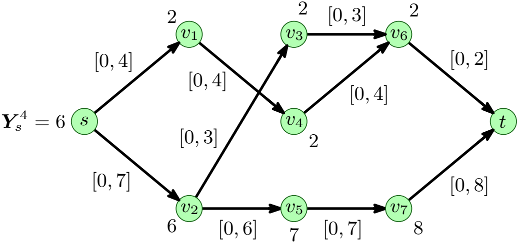
3. Clean-up to restore flow conservation.



$\boldsymbol{Y}_s^4 = 6$
s
v_1
v_2
v_3
v_4
v_5
v_6
v_7
t
2
2
2
2
6
7
8
$[0,4]$
$[0,7]$
$[0,4]$
$[0,3]$
$[0,6]$
$[0,3]$
$[0,4]$
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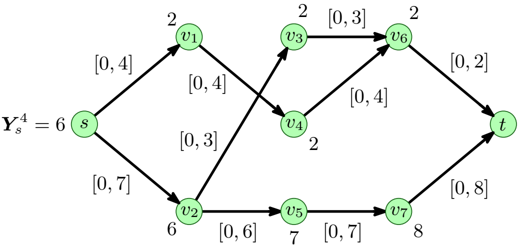
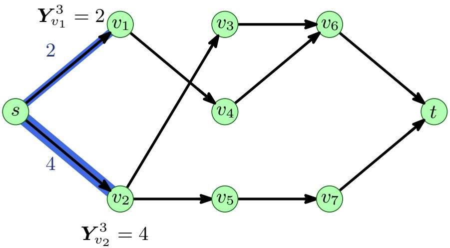
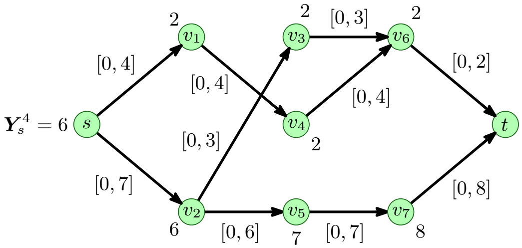
$[0,2]$
$[0,8]$

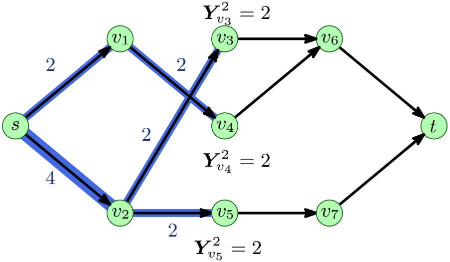


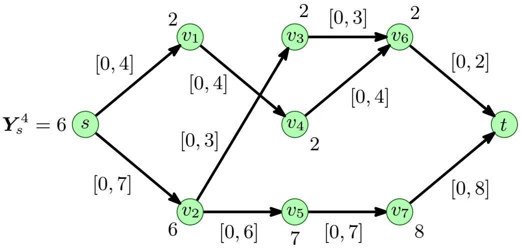
push phase; after $i = 4$



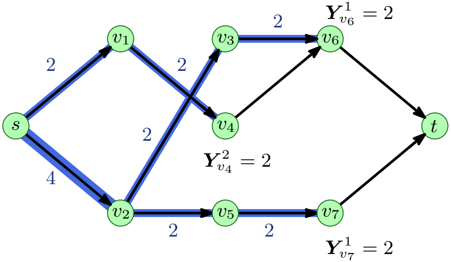


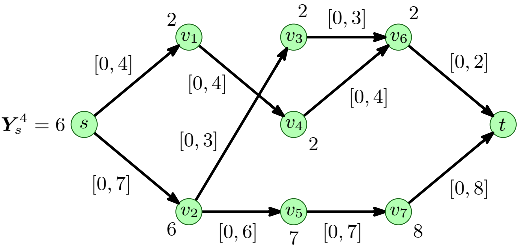
push phase; after $i = 3$



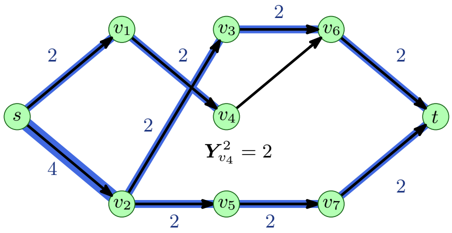


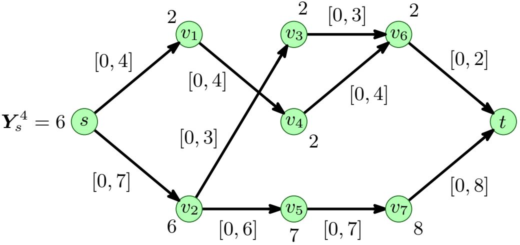
push phase; after $i = 2$



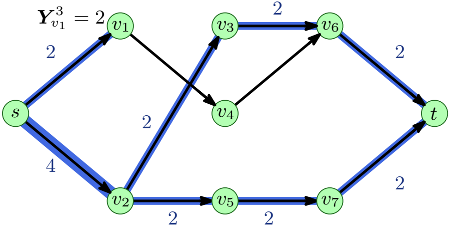


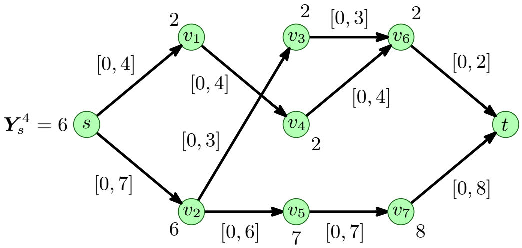
push phase; after $i = 1$





clean-up; after $i = 2$





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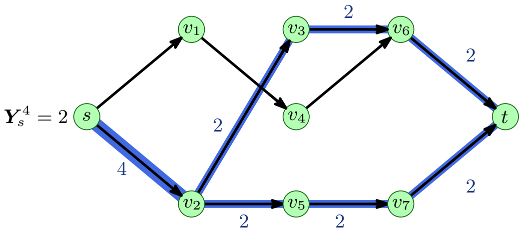

Outlook

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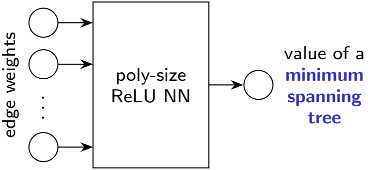

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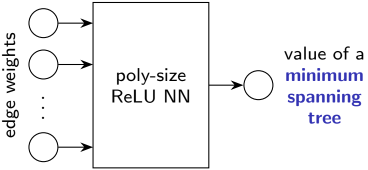

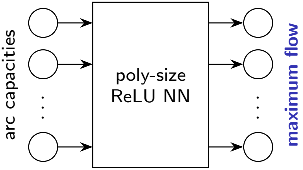

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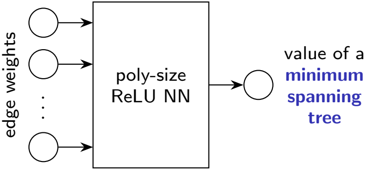

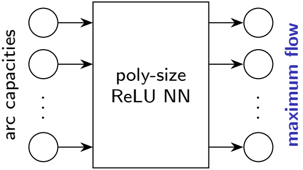

- Which functions can be represented by poly-size NNs?

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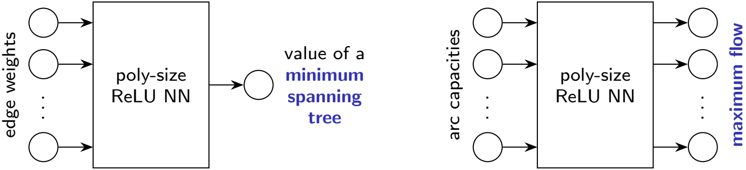

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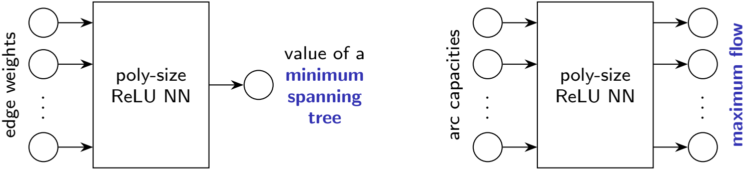

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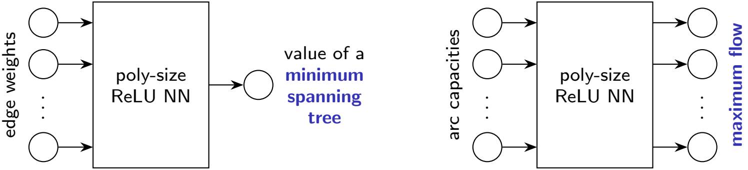

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- Might extended formulations help?